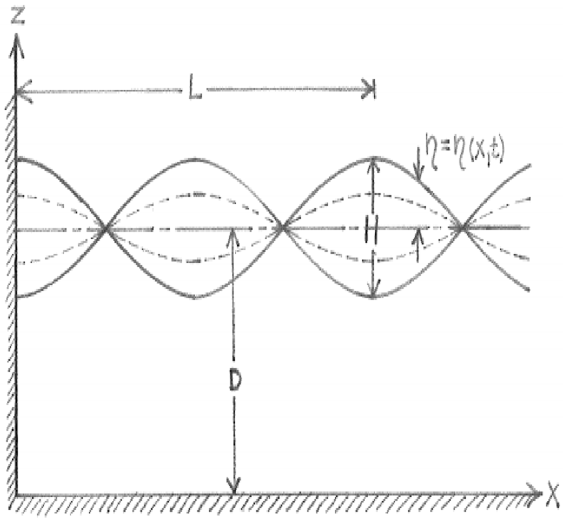
Wave theory - basic principle of development



$\eta = H/2 \cos(\omega t) \cos(kx)$

$p/\gamma = D - z + \eta \cosh(kz) / \cosh(k\mathbf{y})$ water pressure
($p^{+}/\gamma = \eta \cosh(kz) / \cosh(k\mathbf{y})$ wave pressure*)

$w = \partial\eta/\partial t \times \sinh(kz) / \sinh(k\mathbf{y})$ vertical water particle velocity

$(L/T)^2 = g/k \times \tanh(kD)$

$q = H/2 \times L/T \times \sin(\omega t) \sin(kx)$

$u = q \times k \times \cosh(kz) / \sinh(k\mathbf{y})$ horizontal water particle velocity

$\mathbf{y = D + \eta}$, where D = mean water depth, so we use $\mathbf{y}$ = actual water depth (instead of D).

Those are the improved formulas for a regular **standing wave**, and they can be proved to be theoretically correct of 1' order (see: http://lavigne.dk/waves/simplewavesformulas.pdf page 9). But did we just guess these formulas with the help of the classical wave solutions (which is OK), or is there some obvious principle to evolve a simple mathematical practical wave theory?

We will now consider how to find those formulas and determine the wave theory.

We start by observing regular nonbreaking waves in the nature and in the laboratory.

At a vertical wall we see the surface of the water move periodically up and down, with the surface velocity $w_s = \partial\eta/\partial t$. The vertical water velocity at the bottom must be $w_b = 0$. So the vertical velocity decreases from $w_s = \partial\eta/\partial t$ at the surface to $w_b = 0$ at the bottom. Decreases with what function?

For infinite deep water it is realistic to propose that this decrease of w by the depth could in some way be exponential, so (with z=0 at MWL) let us try with the expression:

$w = w_s \; e^{R(z-\eta)}$

With the conservation of mass equation:

$\partial u/\partial x + \partial w/\partial z = 0$; so $\partial u/\partial x = -R\, w_s\, e^{R(z-\eta)}$

then we get that the horizontal velocity is also decreasing exponentially by the depth.

If the water is not so deep we propose the exponential functions to change to hyperbolic functions. So with $w_b = 0$ at the bottom we propose sinh(Rz) (with z=0 at the bottom) and we get:

$w \approx \partial\eta/\partial t \sinh(Rz)$; or: $w = \partial\eta/\partial t \times \sinh(Rz) / \sinh(Ry)$ because of $w = \partial\eta/\partial t$ for $z = y = D + \eta$

then with the conservation of mass equation we get that the horizontal velocity is vertically distributed as a cosh(Rz) function.

From observing a regular **progressive wave** in a laboratory wave flume tank, or look on a drawing (next page), then it looks like the whole crest of the wave above MWL is sliding with the wave celerity c on the MWL giving that the water discharge q through a vertical is:

$q = c\,\eta$

so then we get, using: $\int_0^y u\,dz = q$:

$u = c\,\eta\,R\,e^{R(z-\eta)}$ for the progressive wave horizontal particle velocity in deep water, and

$u = c\,\eta\,R \cosh(Rz)/\sinh(Ry)$ for the non deep water progressive wave

From our sketches (next page) we see that:

$\partial q/\partial x = -\partial\eta/\partial t$ and $-\partial\eta/\partial t = c\,\partial\eta/\partial x$

1. So starting with the proposal for the horizontal particle velocity u:

2. we use the conservation of mass equation to get the vertical particle velocity w:

3. then use the vertical dynamic equation (Newton's 2' law) to get the water pressure p

4. then use the horizontal dynamic equation to get a wave equation

5. this wave equation is split into a z-dependent equation

6. and a z-independent equation

7. neglecting (some of) the theoretical negligible 2' and higher order terms we get the wave solution.

(Some definition sketches on the next page)

Progressive 1' order wave formulas:

$\eta = H/2 \cos(k(x - ct))$

$k = 2\pi/L$

$c^2 = (L/T)^2 = g/k \times \tanh(kD)$

$u = c\,\eta\,k \cosh(kz)/\sinh(ky)$ horizontal water particle velocity

$w = \partial\eta/\partial t \times \sinh(kz)/\sinh(ky)$ vertical water particle velocity

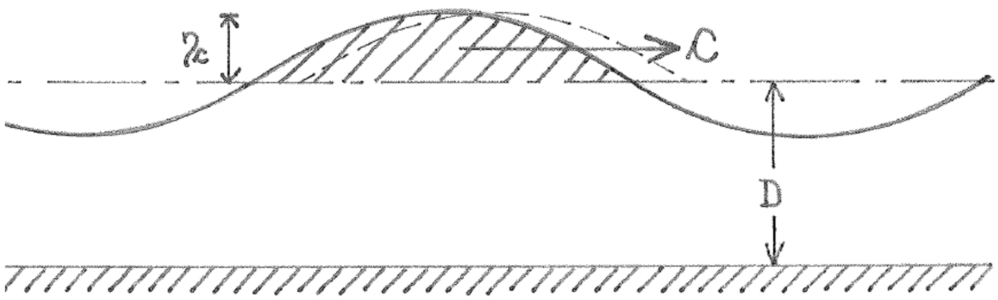
$p/\gamma = D - z + \eta \cosh(kz)/\cosh(ky)$ water pressure ($p^+/\gamma = \eta \cosh(kz)/\cosh(ky)$* below MWL and trough)

$y = D + \eta$, where D = mean water depth, so: y = actual water depth

Mean water level, MWL, z=D: pressure $p^+/\gamma = \eta \cosh(kD)/\cosh(k(\mathbf{D+\eta}))$; (NOT Airy: $p^+/\gamma = \eta$)

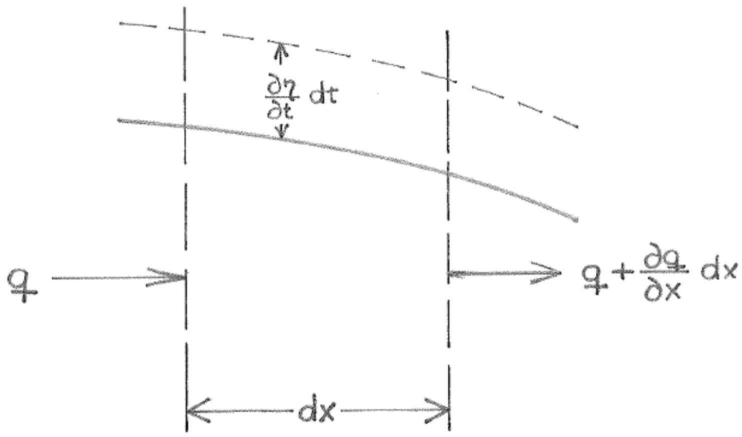
Surface of the trough: p = 0 so $p^+/\gamma = \eta$ (negative); (NOT Airy: $p^+/\gamma = \eta \cosh(kz)/\cosh(kD)$)

(The famous classical Airy wave (a mathematical potential theory) gives: $p^+/\gamma = \eta$ at MWL. This is mathematically within a 1' order approximated wave theory the same as $p^+/\gamma = \eta \cosh(kD)/\cosh(k(D+\eta)$, but not in realistic numerical practice.)

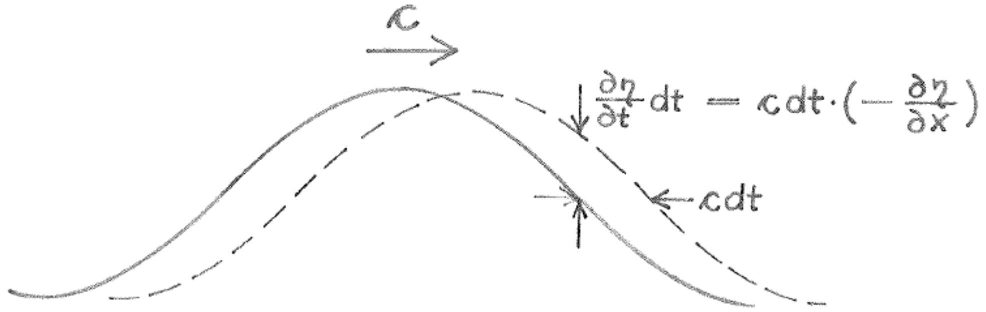


When observing progressive waves of permanent form it looks as if the shown part of the crest slides on 'frozen' water. The trough is considered as a sliding negative crest. In this way the water discharge through a vertical is determined.

$q = c\,\eta$



The equation of continuity for the water discharge, q.



A progressive wave of constant form.

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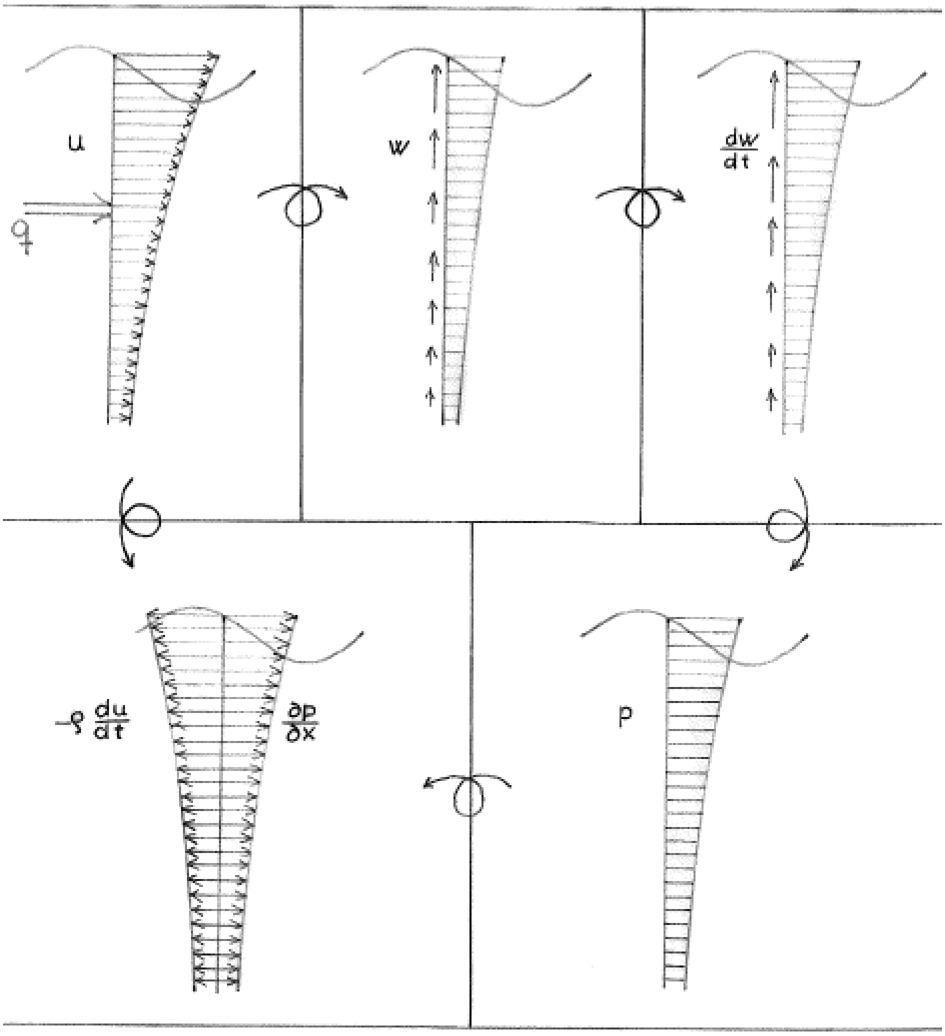
Fig. 12. THE BASIC PHYSICAL PRINCIPLES OF THE WAVE THEORY.



The horizontal velocity u is written as an unknown function.
The equation of continuity gives the vertical velocity w.
w and u give the vertical acceleration dw/dt.
The vertical equation of momentum then gives the pressure p.
This gives the horizontal pressure gradient ∂p/∂x.
u and w also give the horizontal acceleration du/dt, or the force of inertia -ρdu/dt.
Finally -ρdu/dt and ∂p/∂x must ballance each other at any point of the fluid, which determines the unknown function of u and the wave profile.

Principle of wave development: From chapter 2 page 49 in my book, see: http://lavigne.dk/waves/wavese.htm , or www.mejlhede.dk and select: Ocean waves and wave pressure

(A direct simple (non potential) wave theory based on observing the water waves when I was performing model tests for wave pressure on a vertical face breakwater in a laboratory wave flume tank).

Solving the wave equation principle

Do not right away drop all the negligible higher order terms!

As shown on the figure on the previous page (page 4), we see that we end up with the horizontal dynamic equation of motion:

$\partial p/\partial x + \rho\ G_x = 0$

This gives us a wave equation of several terms of different importance:

A + B + C + D + E + p + q + r + s = 0

Some terms are of 1' order magnitude, i.e. involving H^1 or η^1, some are of 2' order H^2, and some are of higher orders. We cannot directly solve this wave equation with all its terms to give us a number of different solutions. So instead we select to solve it in e.g. 3 parts:

1. A + B = 0; 2. C + D + E = 0; 3. p + q + r + s = 0

and each of the 3 parts then have to be = 0 at the same time t and at the same place x,z all over:
1. a z-dependent equation, 2. a z-independent equation, and a disregarded equation of so called small
3. negligible higher order terms: p + q + r + s. In the traditional wave theory when getting a higher order term it is then right away deleted, e.g. so that **+η** is deleted in the vertical velocity:

$w = \partial\eta/\partial t \times \sinh(kz) / \sinh(k(D\boldsymbol{+\eta}))$ to give: $w = \partial\eta/\partial t \times \sinh(kz) / \sinh(kD)$

which is the traditional Airy expression, but where the surface water particle then does not precisely follow the surface movement $\partial\eta/\partial t$. So in our wave theory here we keep **η** (despite the 2' order) in $\sinh(k(D\boldsymbol{+\eta}))$.

So I propose we keep some of the relevant negligible higher order terms to get a better solution, like get the pressure p=0 at the surface. If a term is deemed mathematically negligible it can never be mathematically wrong to include this term or any other higher order term in our expressions. The well-known 1' order wave theory demands small wave heights, but harbor engineers typically use the 1' order formulas for also the big design waves, so the 1' order formulas should include obvious higher order term corrections.

We could solve the whole wave equation at the surface, $z = D + \eta$ to gain some information (like the celerity c), but then we have to assure that the complete solution is valid all over, for all z, not just for the surface.

So the procedure of this wave formulas development told in 3 lines:

By observing the water of the regular wave by a vertical wall move up and down with a possible vertical sinh distribution we get that the horizontal velocity u is vertically cosh distributed, and with this assumption for u we evolve forward with Newton's 2' law to get a wave equation, and get the wanted formulas.

Niels Mejlhede Jensen, 2022.11.30