CHAPTER IX

PROGRESSIVE CNOIDAL WAVES ON ARBITRARY DEPTH

ABSTRACTS

In this chapter there is developed a finite amplitude cnoidal wave which can be used on any water depth.

This cnoidal wave will have the deep water cnoidal wave as one limit, and the shallow water cnoidal wave and solitary wave as another limit, and the sinusoidal wave as the limit for small wave heights.

Previously it was a problem that harbours often are placed on water depths that for the most common waves are too deep for the shallow water cnoidal waves to be used and too shallow for the Stokes' second order waves. This problem is solved with the second order wave of this chapter which has a surface profile that is described with the same cnoidal function for any water depth. This makes it possible to follow a wave from infinite depth and until just before it breaks, and express the gradual change in surface profile, velocities, pressure, etc.

INTRODUCTION

In chapter VII we saw that the second order sinusoidal wave may not always be so good in practice. The cnoidal wave of chapter VIII could only be used for shallow waters. So there is a need for a new second order theory.

The basic theory of chapter IV, that led to the first order progressive wave on arbitrary depth, and the theory of chapter VI that led to the deep water cnoidal wave, will here be continued to give the progressive cnoidal wave on arbitrary depth.

The final formulas will be checked to be reasonable.

BASIC EQUATIONS

We consider two dimensional progressive waves of permanent form. The bottom is horizontal.

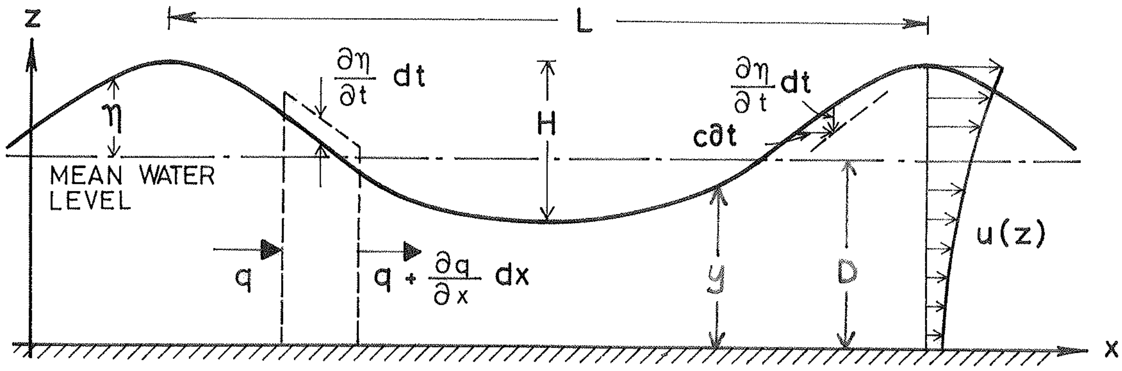


Fig. 1. Definition sketch

From the definition sketch we see that

$$y = D + \eta \tag{1}$$

$y = y(x,t)$ is the actual water depth, D is the mean water depth, and $\eta = \eta(x,t)$ is the surface elevation.

The equation of continuity gives

$$\frac{\partial q}{\partial x} = -\frac{\partial y}{\partial t} = -\frac{\partial \eta}{\partial t} \tag{2}$$

where $q = q(x,t)$ is the water discharge through a vertical.

Further we have for a progressive wave

$$\frac{\partial \eta}{\partial t} = -c\frac{\partial \eta}{\partial x} \tag{3}$$

c is the celerity. Eqs. 2 and 3 give for a wave without a resultant discharge

$$q = c\eta \tag{4}$$

q can also be found by the integration of the horizontal velocity $u = u(x,z,t)$

$$q = \int_0^y u\, dz \tag{5}$$

Like in chapter VII we will write the unknown vertical distribution of u as

$$u = \sum q_i R_i \frac{\cosh R_i z}{\sinh R_i y} \tag{6}$$

and then we will investigate only one of the terms

$$u = q R \frac{\cosh Rz}{\sinh Ry} = c\eta R \frac{\cosh Rz}{\sinh Ry} \tag{7}$$

For the first order wave, R was a constant $R = 2\pi/L = k$. For the second order deep water wave we also had $R = k$. But for the second order sinusoidal wave on arbitrary depth R was partly $R = k$ and partly $R = 2k$, using two terms from eq. 6, as shown in fig. 2 of chapter VII. η_1 and η_{2a} has $R = k$ and only the small second order η_{2b} has $R = 2k$, so writing the total u as only one term, eq. 7, R should be only slightly different from k in a second order wave.

To show the dependence on the wave height we could write

$$R = k + \beta k^2 H \tag{8}$$

But we can make a better proposal. If we wanted u in fig. 2 of chapter VII to be written only as one term in eq. 7 with a cosh Rz distribution, then it is seen that R should be slightly bigger than k for the crest and slightly less than k for the trough. This makes us propose

$$R = k(1 + \beta k \eta) \tag{9}$$

instead of eq. 8. For deep water waves we would then find $\beta = 0$. We could also combine the information of eqs. 8 and 9 and write

$$R = r(1 + \beta k \eta) \quad \text{or} \quad R = r + \beta k^2 \eta \tag{10}$$

where $r = r(H)$ so that

$$r \rightarrow k \quad \text{for} \quad kH \rightarrow 0 \tag{11}$$

and so that r 'differs from k only with a term an order smaller'.

This means that we may also write

$$R = r(1 + \beta \frac{k^3}{r^2} \eta) \tag{12}$$

which makes the dependence of R on η less for bigger wave heights.

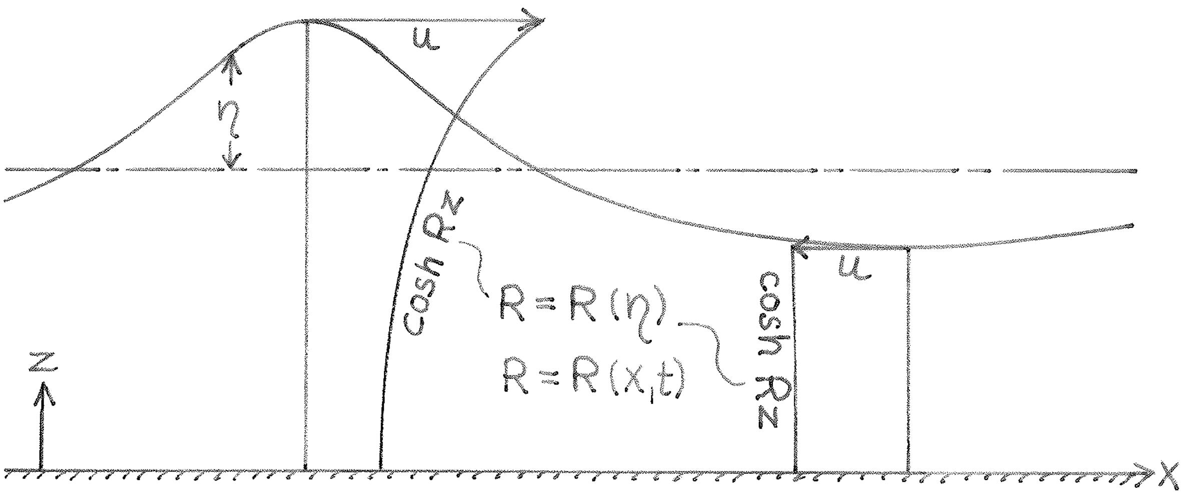


Fig. 2 In a second order wave we now want to express u as only one term with a cosh Rz distribution. This means that R must be a function of x,t instead of just a constant.

Here in the beginning of the development of the wave theory we could give a lot more proposals for R, but for some time we only need to be aware of, that $R = R(x,t)$ in the general case, and that R is a constant for infinite deep water, and we can use either eq. 9, 1o, or eq. 12. (The final expression for R will still be changed at the end, see the appendix.)

Through the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{13}$$

and with the condition $w = 0$ at the bottom $z = 0$ we get for the vertical velocity $w = w(x,z,t)$

$$w = c\frac{\partial \eta}{\partial x}[-1+\eta R \coth Ry]\frac{\sinh Rz}{\sinh Ry} - c\eta\frac{\partial R}{\partial x}\frac{z\cosh Rz - y\coth Ry \sinh Rz}{\sinh Ry} \tag{14}$$

With the new x-dependent R we got an extra term with $\frac{\partial R}{\partial x}$ in the expression for w. We see that the term is of second order magnitude. With our experience from the previous three chapters of developing second order theories, we know that whenever wanted substitutions with the first order expressions given later in eq. 26 is permissible in the second order terms. For infinite deep water we have $\frac{\partial R}{\partial x} = 0$, so that the extra term vanishes. We see that at the bottom $z = 0$, and at the surface $z = y$ the term will always be 0. We shall now show that otherwise the term can also be neglected.

For shallow water first order sinusoidal waves certain approximations can be used, such as

$$R\,z \cosh R\,z \cong \sinh R\,z \tag{15}$$

$$R\,y \coth R\,y \cong 1 \tag{16}$$

$$R\,z \sinh R\,z \cong 2[\cosh R\,z - 1] \tag{17}$$

This can be seen out from the Maclaurin series for $R\,y = k\,y \rightarrow 0$ or $D/L \rightarrow 0$. For our problem here these approximations can only be used in second order terms in a shallow water theory. By this it is seen that the last term in eq. 14 will be zero in a second order theory. This approximation is considered further in the appendix.

So for w we end up with only

$$w = c\frac{\partial \eta}{\partial x}[-1 + \eta R \coth Ry]\frac{\sinh Rz}{\sinh Ry} \tag{18}$$

The vertical particle acceleration $G_z = G_z(x,z,t) = \frac{dw}{dt}$ is then found from eqs. 18 and 7, and then the vertical dynamic equation

$$-\frac{\partial p}{\partial z} - \gamma = \rho G_z \tag{19}$$

gives us the pressure $p = p(x,z,t)$ by integration

$$\frac{p}{\gamma} = y - z + \frac{c^2}{g}\left\{\left[\frac{\partial^2\eta}{\partial x^2}\frac{1}{R} - \left[2\left(\frac{\partial\eta}{\partial x}\right)^2 + \eta\frac{\partial^2\eta}{\partial x^2}\right]\coth Ry\right] \cdot \frac{\cosh Ry - \cosh Rz}{\sinh Ry} + \frac{1}{4}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry}\right\} \tag{20}$$

γ is the unit weight, ϱ is the unit mass, g is the acceleration of gravity, so $\gamma = \varrho g$. The constant of integration was p = o at the surface z = y. In eq. 2o we got an extra term with $\frac{\partial R}{\partial x}$ like in eq. 14, but again the term could be neglected.

It is though not possible to continue to have all the terms with $\frac{\partial R}{\partial x}$ neglected. For the horizontal particle acceleration G_x = $G_x(x,z,t)$ we find

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} =$$
$$c^2\left\{-\frac{\partial \eta}{\partial x}R\frac{\cosh Rz}{\sinh Ry} + \eta\frac{\partial \eta}{\partial x}R^2\frac{\cosh Ry\cosh Rz + 1}{\sinh^2 Ry}\right.$$
$$\left.-\eta\frac{\partial R}{\partial x}\frac{2[\cosh Rz - 1]}{\sinh Ry}\right\} \qquad (21)$$

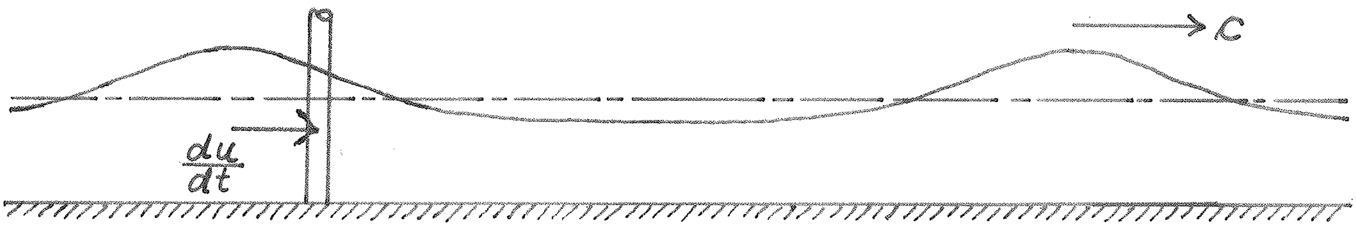


Fig. 3 In calculating wave forces on piles it is of interest to consider the horizontal particle acceleration G_x = du/dt in eq. 21. The second order term with $\partial R/\partial x$ can then in practice usually be neglected. For deep water waves we have $\partial R/\partial x$ = o. With R = k in a second order term we have that for shallow water cosh kz is close to 1. And further we have that usually G_x is of importance when η is not so big. So for G_x we can use the usual expression, only with a new value for R, just like we could use the usual expressions for u, w, and p.

Differentiating eq. 2o we can get the following expression for $\partial p/\partial x$

$$\frac{1}{\gamma}\frac{\partial p}{\partial x} = \frac{\partial \eta}{\partial x} + \frac{c^2}{g}\Big\{\Big[\frac{\partial^3\eta}{\partial x^3}\frac{1}{R} - \Big[6\frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} + \eta\frac{\partial^3\eta}{\partial x^3}\Big]\coth Ry\Big]$$
$$\cdot\frac{\cosh Ry - \cosh Rz}{\sinh Ry} + \frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}$$
$$+\frac{1}{4}\Big[\frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} - \eta\frac{\partial^3\eta}{\partial x^3}\Big]\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry}$$
$$-\frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\frac{\cosh Ry - \cosh Rz}{\sinh Ry}$$
$$+\frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}(Ry \coth Ry)\tanh Ry - \frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\frac{Rz \sinh Rz}{\sinh Ry}$$
$$-\frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\frac{Ry \coth Ry(\cosh Ry - \cosh Rz)}{\sinh Ry}\Big\} \qquad (22)$$

The $\partial R/\partial x$ terms of eq. 22 can be reduced to, using eqs. 16 and 17

$$-2\frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\coth Ry + \frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\tanh Ry$$
$$+2\frac{\partial^2\eta}{\partial x^2}\frac{1}{R^2}\frac{\partial R}{\partial x}\frac{1}{\sinh Ry} \qquad (23)$$

Through the horizontal equation of momentum

$$-\frac{\partial p}{\partial x} = \rho G_x \qquad (24)$$

we find another expression for $\partial p/\partial x$. Eliminating $\partial p/\partial x$ from the two equations we get the wave equation

$$\frac{g}{c^2}\frac{\partial \eta}{\partial x} + \frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\frac{\cosh Ry - \cosh Rz}{\sinh Ry} - \frac{\partial \eta}{\partial x}R\frac{\cosh Rz}{\sinh Ry}$$
$$+\frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} - \Big[6\frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} + \eta\frac{\partial^3\eta}{\partial x^3}\Big]\frac{\coth Ry[\cosh Ry - \cosh Rz]}{\sinh Ry}$$
$$+\eta\frac{\partial \eta}{\partial x}R^2\frac{\cosh Ry \cosh Rz + 1}{\sinh^2 Ry} + \frac{1}{4}\Big[\frac{\partial \eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} - \eta\frac{\partial^3\eta}{\partial x^3}\Big]\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry}$$
$$-2\frac{\partial^2\eta}{\partial x^2}\frac{\partial R}{\partial x}\frac{\cosh Ry - 1}{R^2 \sinh Ry} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial R}{\partial x}\frac{1}{R^2}\tanh Ry$$
$$-2\eta\frac{\partial R}{\partial x}\frac{\cosh Rz - 1}{\sinh Ry} = 0 \qquad (25)$$

In the first order terms it is important to remember that now R is variable, so that when using eq. 10 we get extra second order terms.

If we keep only the three first order terms from eq. 25 we get the first order solution

$$\eta = \eta_1 = \frac{H}{2} \cos k(x - ct) \qquad R = k = \frac{2\pi}{L} \tag{26}$$

THE CNOIDAL SOLUTION

When the second order wave equation was used in chapter VII to find the second order sinusoidal solutions, the first order solution, η_1 had been inserted in all the second order terms of the wave equation.

However this time some of the second order terms in the wave equation, eq. 25, are only approximated in view of the first order solution. For instance eq. 26 gives

$$\frac{\partial^2 \eta}{\partial x^2} = -k^2 \eta \quad ; \quad \frac{\partial^3 \eta}{\partial x^3} = -k^2 \frac{\partial \eta}{\partial x} \tag{27}$$

so that e.g. the following substitutions can be made in second order terms

$$\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} \simeq \eta \frac{\partial^3 \eta}{\partial x^3} \; ; \; -\eta \frac{\partial \eta}{\partial x} R^2 \simeq \eta \frac{\partial^3 \eta}{\partial x^3} \simeq \frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} \tag{28}$$

Further R = k and y = D can be used when needed in second order terms.

The wave equation, eq. 25, is solved by splitting it into a z-dependent equation and a z-independent equation. The z-dependent equation will be

$$\begin{aligned}&\Big[-\frac{\partial^3 \eta}{\partial x^3}\frac{1}{R} - \frac{\partial \eta}{\partial x} R + \Big[6 \frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} + \eta \frac{\partial^3 \eta}{\partial x^3} + \eta \frac{\partial \eta}{\partial x} R^2\Big] \coth Ry \\ &- 2\eta \frac{\partial R}{\partial x}\Big] \frac{\cosh Rz}{\sinh Ry} \\ &- \frac{1}{4}\Big[\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} - \eta \frac{\partial^3 \eta}{\partial x^3}\Big] \frac{\cosh 2Rz}{\sinh^2 Ry} = 0\end{aligned} \tag{29}$$

With eq. 28 we see that the term with $\cosh 2Rz$ will vanish. Remembering that r can be substituted by k in second order terms, eq. 29 will reduce to, after some calculations using eqs. 1o and 28,

$$\frac{\partial^3\eta}{\partial x^3} = -\frac{\partial\eta}{\partial x}R\left[R + 6\eta k^2 \coth kD + 2\beta\eta k^2\right]$$
$$= -\frac{\partial\eta}{\partial x}\left[r^2 + 6\eta k^3 \coth kD + 4\beta\eta k^3\right] \qquad (3o)$$

It will now be shown that a solution to this equation is

$$\eta = H\,cn^2\,\frac{2K}{L}(x-ct) + \eta_t \qquad (31)$$

with the negative trough depth η_t

$$\eta_t = \frac{H}{m}\left(1 - m - \frac{E}{K}\right) \qquad (32)$$

where $K = K(m)$ and $E = E(m)$ are the complete elliptic integrals of first and second kind and m is the parameter = the square of the modulus.

η_t is found from the definition of the mean water level by the integration

$$\int_0^L \eta\,dx = o \qquad (33)$$

It is known from chapter VIII that for small waves

$$cn \rightarrow \cos, \quad K \rightarrow \frac{\pi}{2}, \quad E \rightarrow \frac{\pi}{2} \quad \text{for } m \rightarrow o \qquad (34)$$

and

$$\eta_t \rightarrow -\frac{H}{2} \quad \text{for } m \rightarrow o \qquad (35)$$

By differentiating eq. 31 once to get $\partial\eta/\partial x$ and three times to get $\partial^3\eta/\partial x^3$ it can be shown purely mathematically that

$$\frac{\partial^3\eta}{\partial x^3} = -\frac{\partial\eta}{\partial x}\,\frac{16K^2}{L^2}\left[1 - 2m + 3m\,cn^2\,\frac{2K}{L}(x-ct)\right] \qquad (36)$$

($\partial\eta/\partial x$ and $\partial^3\eta/\partial x^3$ are given in chapter VIII.)

This expression is inserted in eq. 3o. The second order terms of eq. 3o, containing $\frac{\partial\eta}{\partial x}\eta$ can be approximated in view of the first order solution, eq. 26, so $\eta_t = -\frac{H}{2}$ from eq. 35 will be used when substituting η in eq. 3o by eq. 31.

Then we get

$$\frac{16K^2}{L^2}\left[1-2m+3m\,cn^2\frac{2K}{L}(x-ct)\right]=$$

$$r^2+6k^3\left(\coth kD+\frac{2}{3}\beta\right)H\left[cn^2\frac{2K}{L}(x-ct)-\frac{1}{2}\right] \tag{37}$$

This equation is split into two equations, one depending on the cn-terms and one depending on the other terms. The cn -dependent equation gives the condition, using $k = \frac{2\pi}{L}$,

$$mK^2=\pi^3\frac{H}{L}\left[\coth kD+\frac{2}{3}\beta\right] \tag{38}$$

The cn-independent equation gives, using eq. 38

$$r=\frac{4K}{L}\sqrt{1-\frac{m}{2}} \tag{39}$$

For $m \to o$ eq. 39 gives $r \to k = \frac{2\pi}{L}$, the first order sinusoidal expression.

We have now showed that the z-dependent wave equation can be fulfilled with η in eq. 31 under the conditions specified in eqs. 32, 38, and 39. But the z-independent equation must also be fulfilled under at least the same conditions.

In a second order theory we can use

$$\coth Ry = \coth RD - \frac{R\eta}{\sinh^2 RD} \tag{4o}$$

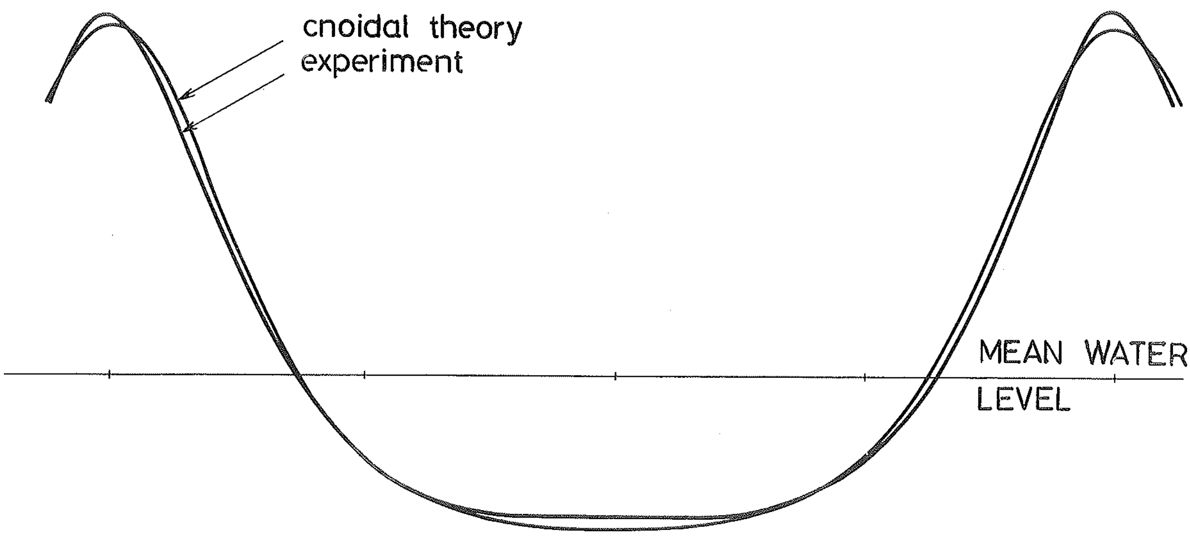
Using eq. 1o we get

$$\coth RD = \coth(r+\beta k^2\eta)D$$

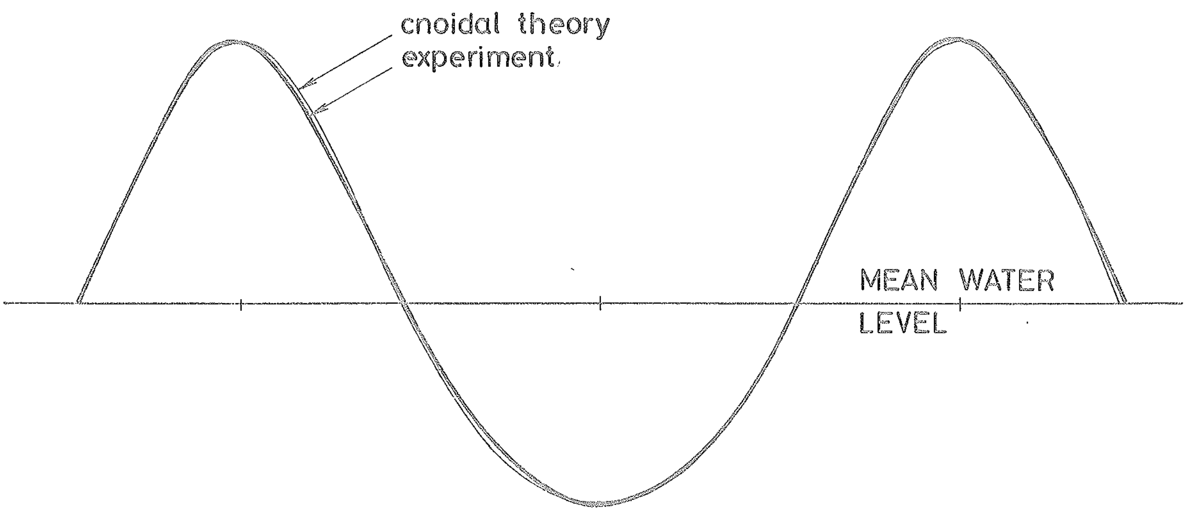
$$= \frac{1+\tanh rD\ \tanh\beta k^2\eta D}{\tanh rD+\tanh\beta k^2\eta D} \tag{41}$$

In view of the Maclaurin series we get, neglecting higher order terms

$$\coth RD = \coth rD + \beta k\eta \tanh kD \tag{42}$$



H/L = 4.3 % , H/D = o.476 , L/D = 11.o



H/L = 9.1 % , H/D = o.178 , L/D = 2.o2

Fig. 4 and 5. Surface profiles of waves passing a wave recorder in an experimental flume compared to the cnoidal theory of this paper.

The z-independent equation from eq. 25 will be

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\coth Ry+\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}$$
$$-\left[6\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}+\eta\frac{\partial^3\eta}{\partial x^3}\right]\coth^2 Ry+\eta\frac{\partial\eta}{\partial x}\frac{R^2}{\sinh^2 Ry}$$
$$+\frac{1}{4}\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^3\eta}{\partial x^3}\right]\frac{\cosh 2Ry}{\sinh^2 Ry}$$
$$-2\frac{\partial^2\eta}{\partial x^2}\frac{\partial R}{\partial x}\frac{\cosh Ry-1}{R^2\sinh Ry}+\frac{\partial^2\eta}{\partial x^2}\frac{\partial R}{\partial x}\frac{1}{R^2}\tanh Ry$$
$$+2\eta\frac{\partial R}{\partial x}\frac{1}{\sinh Ry}=0 \tag{43}$$

$\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}$ is substituted by eq. 30 with eq. 10 and using eqs. 28, 40, and 42 we get to the second order after reduction and division by $\partial\eta/\partial x$.

$$\frac{g}{c^2}-r\coth rD+\eta k^2\frac{3}{\sinh^2 kD}-2\beta\eta k^2\tanh kD=0 \tag{44}$$

This equation is split into an η-dependent and an η-independent equation to give the further conditions

$$\beta=\frac{3\coth kD}{2\sinh^2 kD} \tag{45}$$

$$c^2=\frac{g}{r}\tanh rD \tag{46}$$

Our second order wave is now determined. We find β from eq. 45 and then mK^2 from eq. 38. From mathematical tables of elliptic functions it is then possible to find m and K and then cn to be used in eq. 31.

It should be emphasized that although the cnoidal solutions given here are correct to the second order, it is possible to find other cnoidal solutions of second order just by making the approximations in the deductions different. This is due to the 'hidden' higher order terms, which can be different in different solutions. It is then of interest to find the best possible second order solutions. Of this reason the second order celerity in eq. 46 will later be changed with third order terms.

DEEP WATER AND SHALLOW WATER LIMITS.

All the final equations given here will for $D/L \to \infty$ give the expressions found for the deep water cnoidal wave in chapter VI. For $D/L \to 0$ we do not get exactly the same wave as the shallow water cnoidal wave of chapter VIII.

DEEP WATER LIMIT.

Eq. 45 gives

$$\beta = \frac{3}{2}\frac{\coth kD}{\sinh^2 kD} \to 0 \quad \text{for } \frac{D}{L} \to \infty \tag{47}$$

Then eq. 38 will be

$$mK^2 = \pi^3 \frac{H}{L}\left[\coth kD + \frac{2}{3}\beta\right] \to \pi^3 \frac{H}{L} \tag{48}$$

which is the same as in chapter VI. Then the rest of the expressions will also be the same as in chapter VI.

SHALLOW WATER LIMIT

For shallow water β in eq. 45 will be

$$\beta = \frac{3}{2}\frac{\coth kD}{\sinh^2 kD} \to \frac{3}{2}\frac{1}{(kD)^3} = \frac{3}{16}\frac{1}{\pi^3}\left(\frac{L}{D}\right)^3 \quad \text{for } \frac{L}{D} \to \infty \tag{49}$$

Then eq. 38 will be

$$mK^2 = \pi^3 \frac{H}{L}\left[\coth kD + \frac{2}{3}\beta\right] = \pi^3 \frac{H}{L}\coth^3 kD$$

$$\to \pi^3 \frac{H}{L}\frac{1}{(kD)^3} = \frac{1}{8}\frac{H}{D}\left(\frac{L}{D}\right)^2 \quad \text{for } \frac{L}{D} \to \infty \tag{50}$$

For the shallow water cnoidal waves of chapter VIII we have for L

$$\left(\frac{L}{D}\right)^2 = \frac{16}{3}\frac{D}{H} mK^2 \tag{51}$$

so

$$mK^2 = \frac{3}{16}\frac{H}{D}\left(\frac{L}{D}\right)^2 \tag{52}$$

So for shallow water there is a small difference between the cnoidal wave of this chapter and the traditional shallow water cnoidal wave. The crest of the cnoidal wave here will be a little less sharp. But this wave will of course have the solitary wave as the extreme limit for $L/D \rightarrow \infty$.

The problems with velocities will be considered later.

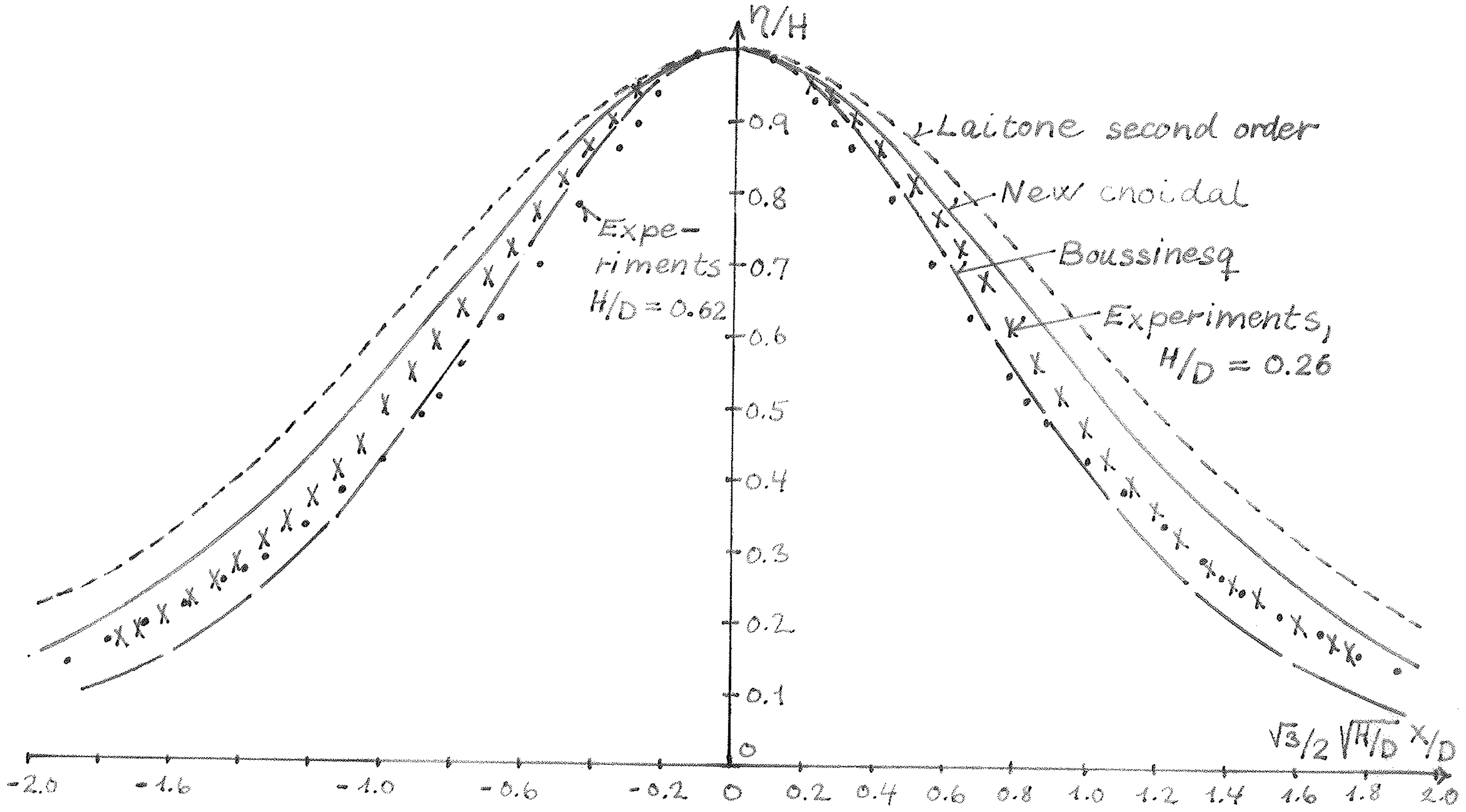


Fig. 6. The solitary wave according to different theories compared with experiments of French, for H/D = o.62 and H/D = o.26. The classical Boussinesq profile is given in chapter VIII, $\eta = H \operatorname{sech}^2 (\sqrt{3}/2 \sqrt{H/D}\, x/D)$. The new profile is the solitary wave limit of the cnoidal wave of this chapter, which gives $\eta = H \operatorname{sech}^2 (1/\sqrt{2} \sqrt{H/D}\, x/D)$. Further the profile of the second order solitary wave of Laitone for H/D = o.62 is shown. As far as the experiments indicate the final stable profile, it is seen that the classical Boussinesq expression is the best, but the difference is not big, especially in consideration of that the new expression as well can be used for deep water waves.

ROTATION

Rotation of second order magnitude in second order waves will be considered.

Rotation and rotational waves could be examined here in the same way as we have done for deep water waves in chapter VI. We would then find that second order waves on arbitrary depth with first order rotation would contain more terms than in the deep water case. But here we will first consider second order rotation.

For the rotation we find

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{1}{2}\left(\frac{H}{2}\right)^2 \omega k^2 \coth kD \frac{\sinh kz}{\sinh kD} \tag{53}$$

where we in this second order term used $R = k$, and $y = D$. So the rotation is a second order constant.

Because the rotation is only a second order expression it is most easy to find it from the sinusoidal second order waves of chapter VII, instead of the cnoidal wave. In the appendix of chapter X it is shown how to find such an expression from the cnoidal wave.

The rotation of eq. 53 can be changed to a different second order value by proposing u in eq. 4 to be

$$u_{rot} = c\eta R \frac{\cosh Rz}{\sinh Ry} + \Delta u\, \delta \tag{54}$$

where

$$\Delta u = c\left(\frac{H}{2}\right)^2 k^2 \coth kD \left[\frac{\cosh kz}{\sinh kD} - \frac{1}{kD}\right] \tag{55}$$

δ is a freely chosen constant, and for $\delta = 1/2$ we get irrotational waves. At the surface Δu in eq. 55 will then be

$$\Delta u_s = c\left(\frac{H}{2}\right)^2 k^2 \coth kD \left[\coth kD - \frac{1}{kD}\right] \tag{56}$$

As this is a second order term in a second order theory it is possible to use either R or r instead of k, if wanted.

VERTICAL DISTRIBUTION OF HORIZONTAL VELOCITY

In this appendix a better expression for R will be given, so that reasonable values for u, w, and p is obtained.

We succeeded all right in finding a new and better wave profile in eq. 31, but we can easily convince ourselves that we did not get any good expressions for u, w, and p with the expressions for R in eqs. 1o or 12 and for β in eq. 45. β will not be changed, but R could as well have been given with a slightly different expression.

The problem is the shallow water limit $D/L \to o$.

For β in eq. 45 we find

$$\beta \to \frac{3}{2}\frac{1}{(kD)^3} \to \infty \quad \text{for } \frac{L}{D} \to \infty \tag{57}$$

From eq. 38 we get

$$mK^2 \to \frac{1}{8}\frac{H}{D}\left(\frac{L}{D}\right)^2 \to \infty \quad \text{for } \frac{L}{D} \to \infty \tag{58}$$

For m we have $o \leq m \leq 1$ and for $m \to 1$ we get $K \to \infty$, so eq. 58 will be

$$K^2 \to \frac{1}{8}\frac{H}{D}\left(\frac{L}{D}\right)^2 \to \infty \quad \text{for } \frac{L}{D} \to \infty \tag{59}$$

Eq. 39 then gives

$$r^2 \to \frac{16K^2}{L^2}\left(1-\frac{1}{2}\right) \to \frac{1}{D^2}\frac{H}{D} \quad \text{for } \frac{L}{D} \to \infty \tag{6o}$$

In eqs. 9 or 1o we assumed that $\beta k \eta \ll 1$, or $\beta k \eta$ is an 'order smaller' than 1. This is sure true for infinite deep water, when $\beta = o$. But at the shallow water limit we get (for a wave that does not vanish), using eq. 57

$$\beta k \eta \to \frac{3}{2}\frac{1}{(kD)^2}\frac{\eta}{D} \to \infty \quad \text{for } \frac{L}{D} \to \infty \tag{61}$$

So the second term in eq. 1o had to be changed. In eq. 12 we proposed

$$R = r\left(1 + \beta \frac{k^3}{r^2}\eta\right) \tag{62}$$

This change of the expression for R will not change the solution to the wave profile because the sinusoidal approximation of the second term will be unchanged

$$\beta \frac{k^3}{r^2}\eta \to \beta k \eta \quad \text{for } m \to o \tag{63}$$

In second order terms we can use the sinusoidal approximations, and the second term of eq. 62 was only involved in second order terms during the development of the wave theory.

From eq. 62 we get using eq. 6o

$$\beta \frac{k^3}{r^2} \eta \rightarrow \frac{3}{2} \frac{1}{(rD)^2} \frac{\eta}{D} \rightarrow \frac{3}{2} \frac{\eta}{H} \qquad \text{for } \frac{L}{D} \rightarrow \infty \tag{64}$$

So we see that by the proposal in eq. 62 the magnitude of the second term is limited. But it is not 'much smaller than 1' for the solitary wave limit, where we at the crest have η = H. So the second term in eq. 62 should still be changed, under the consideration that we still get the same sinusoidal limit as in eq. 63. Rather arbitrary we can propose

$$R = r\left(1 + \frac{1}{5} \tanh 5\beta \frac{k^3}{r^2} \eta\right) \tag{65}$$

which is a correct expression in the second order cnoidal wave of this paper. For the sinusoidal limit, β k η small, we get

$$\frac{1}{5} \tanh 5\beta \frac{k^3}{r^2} \eta \rightarrow \beta k \eta \qquad \text{for } m \rightarrow 0 \tag{66}$$

The maximum value of the second term in eq. 65 is seen to be 1/5 of the first term. So eq. 65 seems reasonable.

There are different reasons why we end up with an expression as eq. 65 instead of eq. 12. We have tried to explain some of them here step by step. The most important is the relationship of eq. 66. This means that we can take eq. 65, go back to eq. 12 and repeat the whole development of the wave with eq. 65 instead of eq. 12, and at the end get exactly the same wave with η of eq. 31, r of eq. 39, and β of eq. 45.

In eq. 65 we have 5 in the argument of tanh and 1/5 as a coefficient. The number 5 has been chosen out from comparison with the expression for u for shallow water cnoidal waves found in chapter VIII. This would not result in quite as simple an expression but the number 5 is a reasonable average.

When an expression like eq. 62 in a second order wave is not fully satisfactory there are different ways of making better expressions. It can be attempted to make a third order theory, which changes all the expressions. But if only one or two expressions need to be improved, they may be found by special calculations that accept other expressions for the second order wave as if they were 'exact'. But the results can still only be claimed to be of second order. We can just illustrate that by finding a different expression for R.

Let us consider the pressure at the bottom, p_b, by substituting $z = o$ in eq. 2o, and then demand that

$$\int_0^L p_b \, dx = \gamma D L \tag{67}$$

For infinite deep water the pressure due to the wave must vanish at big depth, which gives

$$r = \frac{\frac{\partial^2 \eta}{\partial x^2}}{-\frac{g}{c^2}\eta + \frac{7}{4}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{5}{4}\eta \frac{\partial^2 \eta}{\partial x^2}} \tag{68}$$

so that r also for deep water depends on η, as also found by the third order sinusoidal wave theory.

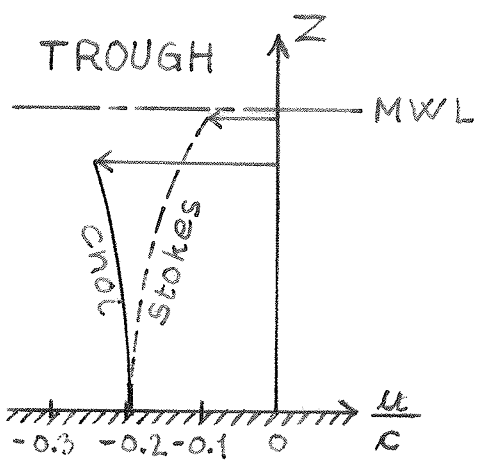
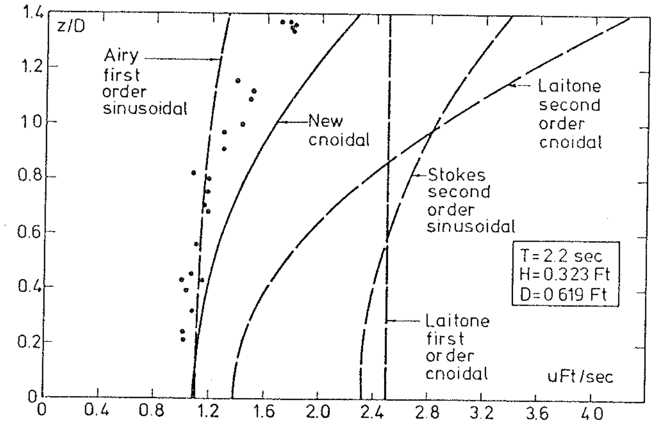
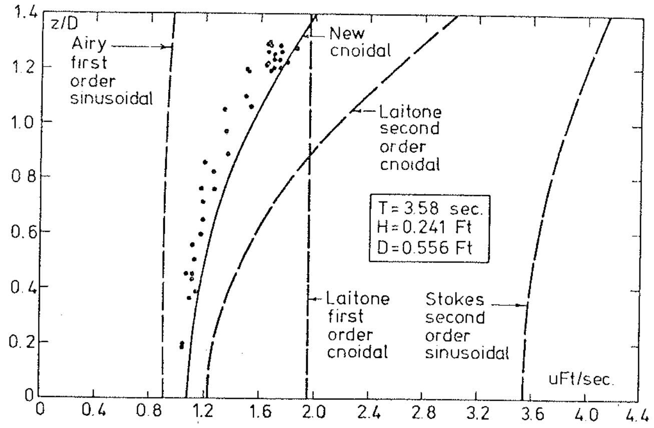


Fig. 7.
Comparison of the horisontal velocity below the middle of the trough according to the Stokes' second order theory and according to the cnoidal theory of this chapter. The wave considered can be : T = 1o.1 seconds, D = 8.o metres, H = 5.4 metres.
The trough depth is rather different according to the two theories, but the velocity at the bottom is close to the same.
The Stokes' velocity profile is rather far from fulfilling the condition, $q = c\eta$, but this does not seem to be a disadvantage in the case here.

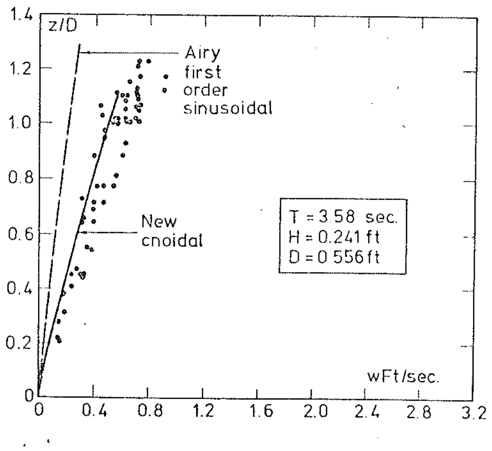


Sinusoidal calculations:

L_o = 7.6 metres
H/D = o.52
L/D = 16
H/L = 3.3%



L_o = 2o metres
H/D = o.43
L/D = 27
H/L = 1.6%



Figs. 8, 9, and 1o.

Maximum horizontal and vertical particle velocities. Comparison of the cnoidal theory of this chapter with other theories and with experiments by Méhauté et al.

CELERITY

In this appendix the celerity will be made dependent on the wave height.

Let us consider how we found the celerity, c. The most easy way to find c in a second order progressive wave is to consider the wave equation at the surface, for z = y. We then find from eq. 25 using $\partial R/\partial x = \beta R^2 \partial\eta/\partial x$

$$\frac{\partial\eta}{\partial x}\left\{\frac{g}{c^2} - R\coth Ry + \frac{\partial^2\eta}{\partial x^2} + \eta R^2 \frac{\cosh^2 Ry + 1}{\sinh^2 Ry} + \beta\frac{\partial^2\eta}{\partial x^2}\tanh Ry\right\} = 0 \qquad (69)$$

\+ neglected third and higher order terms

By selection η and R (and then β) to be of the right combination, as for the second order sinusoidal wave solution or the more complicated for the cnoidal wave solution we may find the simple expression

$$c = \sqrt{\frac{gL}{2\pi}\tanh kD} \qquad (70)$$

But it is also understood that this is only one of the many possible expressions. By including some third and higher order terms or by making different correct substitutions in the second order terms of the type in eq. 28, or in the higher order terms we may get any celerity of the type

$$c^2 = \frac{g}{k}\tanh kD\,[1 + aH^2] \qquad (71)$$

where a is a constant, dependent on D/L. (This type of celerity is found for the third order wave of chapter XI). For any chosen a the above expression will be correct of second order and fit in with the second order theory above. Then it is a question what value should be chosen for a. Usually a is put to a = o in second order theories, but other proposals will be just as correct.

So we can say that the second order expressions eqs. 46 or 7o do not give us a definite value. The second order expression only gives an interval within which the definite value should be found. This interval can be narrowed by making a third order theory. We can also select a specific expression within the second order interval, out from some special conditions we would like to have fulfilled. This is what we will do here.

We will end up by writing c somewhat different from eq. 71, but basically it will be the same, i.e. it will only have a third order difference from eq. 7o.

We could choose c rather arbitrary out from our knowledge of the third order deep water wave, which for an irrotational wave has got, eq. 52 in chapter VI,

$$c_3 = \sqrt{\frac{g}{k}}\sqrt{1+\pi^2\left(\frac{H}{L}\right)^2} \tag{72}$$

and from the classical solitary wave (chapter VIII)

$$c_s = \sqrt{g(D+H)} \tag{73}$$

We could then propose the correct second order expression

$$c = \sqrt{\frac{g}{k}\tanh kD + g(2\eta_c - H)} \tag{74}$$

By this expression we would get eqs. 72 and 73 for $L/D \to o$ and $L/D \to \infty$ as wanted. But for L/D in between we seem to get too big values from eq. 74, so that it would be more reasonable to use the classical sinusoidal expression, eq. 7o.

We will now try to find the celerity using the Bernoulli-equation on the surface of our second order cnoidal wave. This procedure may not always be advisable, because it is based on rather little information about the wave, here only the elevation and the kinematics of the surface. But it seems to give reasonable results.

Let us consider a permanent wave that moves with the celerity c. To make a desired rotation the water has a stream Δu at the surface. Then the particle velocities at the surface of the crest and the trough are $u_c + \Delta u$ and $u_t + \Delta u$ where u_c and u_t are given by eq. 7 and Δu e.g. by eqs 55 or 56.
The hydrodynamic problem is made stationary by regarding the wave in a co-ordinate system that moves with the celerity. Then the Bernoulli equation can be used at the surface at the crest and the trough to give

$$\eta_c + \frac{1}{2g}(c - u_c - \Delta u)^2 = \eta_t + \frac{1}{2g}(c - u_t - \Delta u)^2 \tag{75}$$

Using eq. 7 and $q = c\eta$ we get

$$\begin{aligned}\frac{gH}{c^2} = {} & \left(1 - \frac{\Delta u}{c}\right)\left[\eta_c R_c \coth R_c y_c - \eta_t R_t \coth R_t y_t\right] \\ & - \frac{1}{2}\eta_c^2 R_c^2 \coth^2 R_c y_c + \frac{1}{2}\eta_t^2 R_t^2 \coth^2 R_t y_t\end{aligned} \tag{76}$$

We will then make the approximation

$$R_t \coth R_t y_t \simeq R_c \coth R_c y_c \tag{77}$$

For infinite deep water eq. 77 reduces to r = r. For shallow water η_t in eq. 76 will be small, so then eq. 77 is not important. With eq. 77, eq. 76 becomes

$$\frac{g}{R_c}\tanh R_c y_c = c^2\left[\left(1 - \frac{\Delta u}{c}\right) - \left(\eta_c - \frac{H}{2}\right)R_c \coth R_c y_c\right] \tag{78}$$

We then get for c

$$c^2 = \left(1 + \frac{\Delta u}{c}\right)\frac{g}{R_c}\tanh R_c y_c + g\left(\eta_c - \frac{H}{2}\right) \tag{79}$$

For an irrotational wave Δu is given by eq. 56. The celerity of eq. 79 has been compared with experiments.

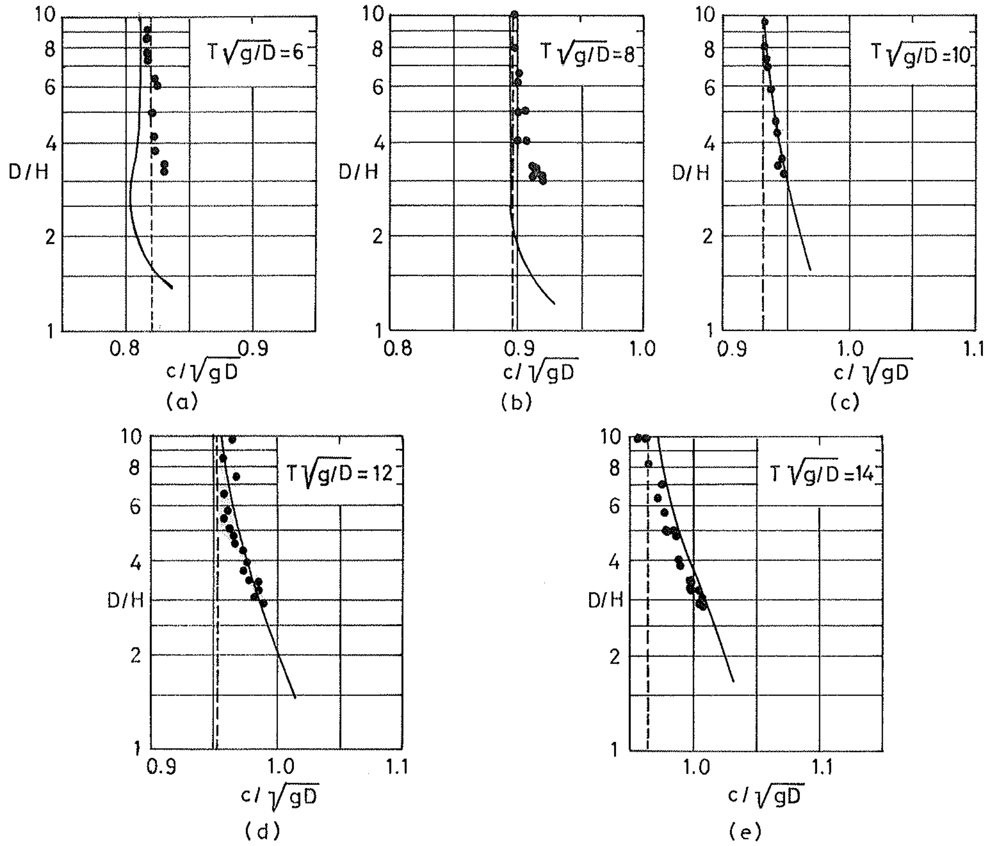


Fig. 11. Comparison of the celerity of the cnoidal theory of this paper and the first order sinusoidal theory with the celerity found in experiments by Tsuchiya and Yamaguchi. For the cnoidal celerity eq. T1o from chapter X has been used. It is seen in some cases to give lower values than the first order theory when experiments show the simple first order theory to be closest to reality. For this reason we have chosen the celerity of the table in chapter X to be the maximum value of eq. 79 and the first order theory.

It is seen that for the realistic situation with waves of T = 1o seconds period and a water depth of D = 1o metres fig (c) is relevant, yielding results with good agreement between theory and experiments.

ENERGY

The potential and kinetic energy of the cnoidal wave will be considered, and the energy flux will be given for calculation of shoaling.

The mean value of the potential energy, E_{pot}, for the cnoidal wave given by eq. 57 is the same as for the traditional shallow water cnoidal wave.

$$E_{pot} = \frac{1}{L}\int_0^L \frac{\gamma}{2}\eta^2 dx = \frac{\gamma}{2L}\int_0^L [H\,cn^2\theta + \eta_t]^2 dx$$
$$= \frac{\gamma}{2L}\left\{H^2\int_0^L cn^4\theta\,dx + 2\eta_t\int_0^L H\,cn^2\theta\,dx + \eta_t^2 L\right\} \quad (80)$$

Which with eqs. 31 and 33 reduces to

$$E_{pot} = \frac{\gamma}{6}\frac{H^2}{m^2}\left[2 - 5m + 3m^2 + 2(2m-1)\frac{E}{K}\right] - \frac{\gamma}{2}\eta_t^2 \quad (81)$$

In eq. 81 it was possible to integrate the surface profile directly, because integrals of cn^2 and cn^4 are known from mathematics. With a perturbation solution like in chapter VII the potential energy would be

$$E_{pot} = \frac{\gamma}{2L}\int_0^L \eta^2 dx = \frac{\gamma}{2L}\int_0^L (\eta_1^2 + 2\eta_1\eta_2 + \eta_2^2)\,dx \quad (82)$$

where the terms are of second, third and fourth order. But if the solution for η had been of third order there would also have been a fourth order term, of $\eta_1\eta_3$. So it is a question if η_2^2 in eq. 82 should be included. As discussed earlier it can never be wrong to include it as long as the energy is not claimed to be calculated with a fourth order precision. It is also seen that when calculating energy for a first order wave the energy will be of second order, and for a second order wave energy terms of third order should be included, (also when calculating energies for shallow water cnoidal waves).

The kinetic energy, E_{kin}, is determined by

$$E_{kin} = \frac{1}{L}\int_0^L\int_0^y \frac{1}{2}\varrho(u^2+w^2)\,dz\,dx \tag{83}$$

Eqs 7 and 18 are substituted into eq. 83. Fourth order terms can be neglected and third order terms can be approximated by the sinusoidal first order solution, also for the cnoidal wave. The first integration gives no problems. The last integration can either be done numerically, or an approximate value can be given analytically which is correct to the third order.

With the bottom $z = o$ as reference level the energy flux through a vertical will be

$$E_{flux} = \int_0^y \left[p + \gamma z + \frac{1}{2}\varrho(u^2+w^2)\right]u\,dz \tag{84}$$

p, u, and w are substituted by eqs. 2o, 7, and 18 to give, after reduction

$$\begin{aligned} E_{flux} = {} & c\gamma D\eta + c\gamma\eta^2 + c^3\varrho\eta\Big\{\frac{\partial^2\eta}{\partial x^2}\frac{1}{2R}\left[\coth Ry - \frac{Ry}{\sinh^2 Ry}\right] \\ & - \left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{1}{2}\eta\frac{\partial^2\eta}{\partial x^2}\right]\coth Ry\left[\coth Ry - \frac{Ry}{\sinh^2 Ry}\right] \\ & + \frac{1}{3}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right] + \frac{1}{2}\eta^2\frac{R^2}{\sinh^2 Ry} + \frac{1}{6}\eta^2R^2 + \frac{1}{6}\left(\frac{\partial\eta}{\partial x}\right)^2 \\ & + \frac{1}{3}\eta\left(\frac{\partial\eta}{\partial x}\right)^2 R\coth Ry + \frac{1}{6}\eta^2\left(\frac{\partial\eta}{\partial x}\right)^2 R^2\coth^2 Ry\Big\} \end{aligned} \tag{85}$$

The last two terms are of fourth and fifth order and can be neglected. The third order terms can be approximated, so that $\eta^2R^2 = -\eta\frac{\partial^2\eta}{\partial x^2}$. The transported energy will then be

$$\begin{aligned} E_{trans} = {} & \frac{1}{T}\int_0^T E_{flux}\,dt = \frac{\gamma}{T}\int_0^L \eta^2 dx + \frac{\varrho c^2}{T}\int_0^L \eta\Big\{\frac{\partial^2\eta}{\partial x^2}\frac{1}{2R} \\ & \cdot\left[\coth Ry - \frac{Ry}{\sinh^2 Ry}\right] + \left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{1}{2}\eta\frac{\partial^2\eta}{\partial x^2}\right]\frac{Ry}{\sinh^2 Ry} \\ & + \left(\frac{\partial\eta}{\partial x}\right)^2\left[\frac{1}{2} - \coth^2 Ry\right] - \eta\frac{\partial^2\eta}{\partial x^2}\coth^2 Ry\Big\}dx \end{aligned} \tag{86}$$

The first term is known from the potential energy, eq. 81. The last integral can as well be calculated numerically as finding an approximate analytical value.

With the calculation of the transported energy it is possible to use the cnoidal theory here for practical assignments with regular waves as shown in the numerical example in chapter X.

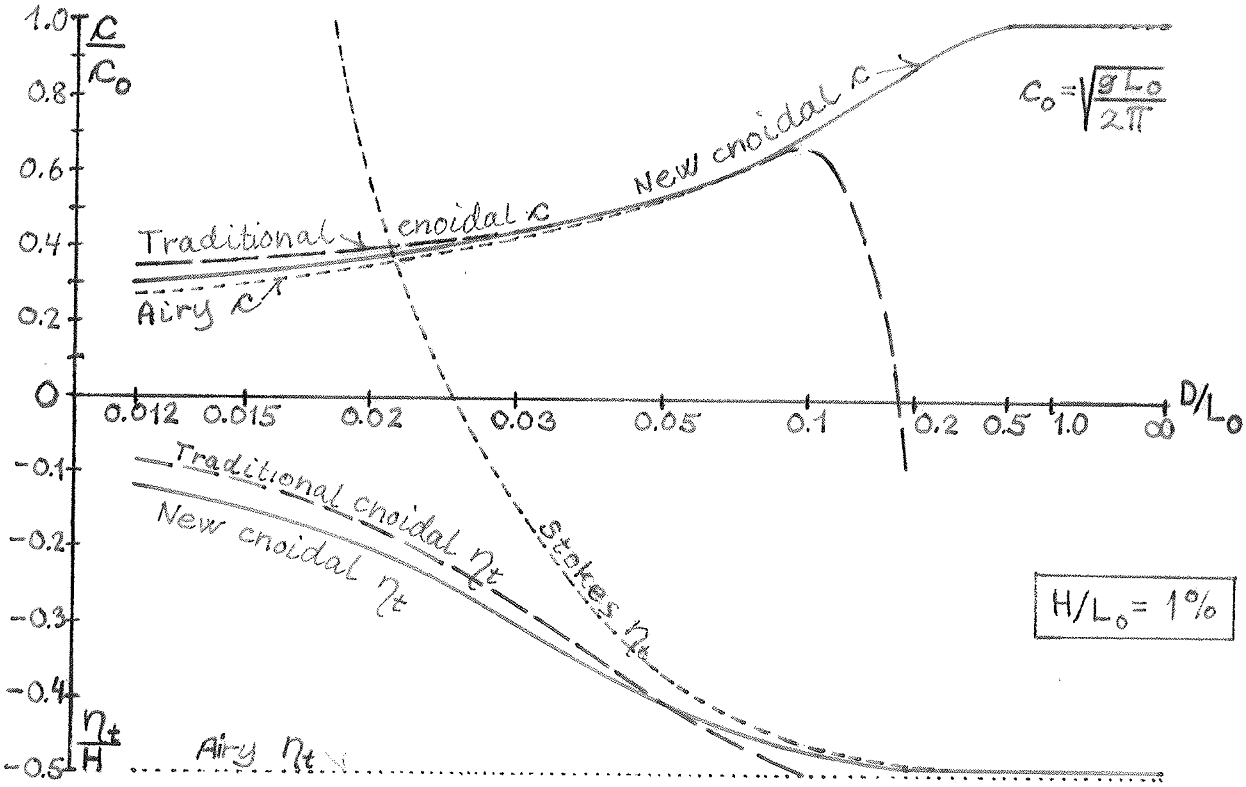


Fig. 12. Comparison of the traditional shallow water cnoidal theory, the Airy theory, and the Stokes' theory with the new cnoidal theory, with respect to the celerity and the trough depth. At the shallow water end the wave height will reach H/D = o.8. While the traditional theories have deficiencies at one end or the other, the new cnoidal theory can be used for any D/L_o.

COMPARISON OF CNOIDAL TERMS FOR THE VERTICAL VELOCITY

The neglected term in the expression for the vertical velocity will here be shown to be negligible. Eq. 14 will be written

$$\frac{w}{c} = -\frac{\partial\eta}{\partial x}\frac{\sinh Rz}{\sinh Ry} + \eta\frac{\partial\eta}{\partial x}R\frac{1}{\sinh Ry}A(z)$$
$$+\eta\frac{\partial\eta}{\partial x}R\frac{1}{\sinh Ry}B(z) \tag{87}$$

where $A(z)$ and $B(z)$ are the functions we want to investigate. They are seen to be

$$A(z) = \coth Ry \sinh Rz \tag{88}$$

$$B(z) = -\frac{\partial R}{\partial x}\frac{1}{R}\Big/\frac{\partial\eta}{\partial x}[z\cosh Rz - y\coth Ry \sinh Rz] \tag{89}$$

For R we ended up with the expression of eq. 65, which gives

$$\frac{\partial R}{\partial x} = \beta\frac{k^3}{r}\frac{\partial\eta}{\partial x}\frac{1}{\cosh^2 5\beta\eta k^3/r^2} \tag{90}$$

so that for the positive values we get

$$\frac{\partial R}{\partial x} \leq \beta k^2\frac{\partial\eta}{\partial x} \tag{91}$$

This expression would also be a result of eqs. 9, 10, or 12. Using eq. 91 in eq. 89 we get that numerically $B(z)$ will not exeed

$$B(z) = -\beta\frac{k}{R}[kz\cosh Rz - ky\coth Ry\sinh Rz] \tag{92}$$

$A(z)$ of eq. 88 and $B(z)$ of eq. 92 are compared in fig. 13. We know from previously (eqs. 9, 10, 12) that R is bigger or smaller than k depending on the part of the wave considered. So for convenience we here consider the situation with $R = k$.

We see that $A(z)$ is more than 10 times bigger than $B(z)$ even for transitional waves, so in eq. 87 (or eq. 14) it is right to neglect the last term as much smaller than the second order term.

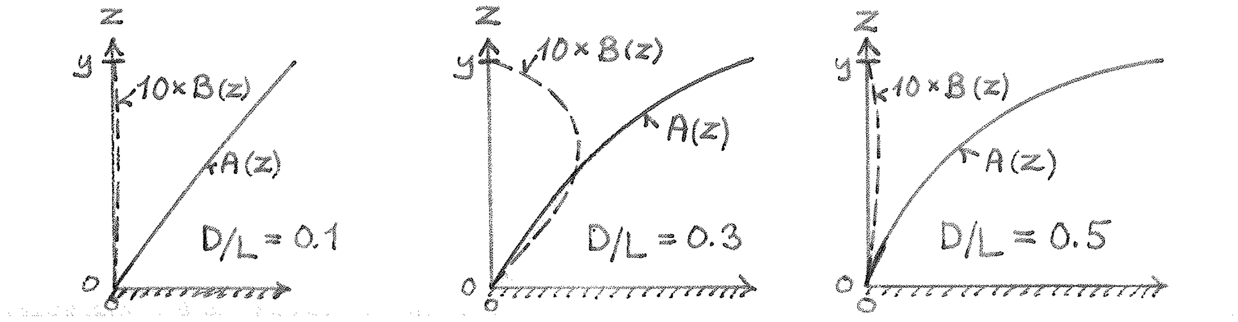


Fig. 13. In the final expression for w (eq.18) a term was neglected (eq.14). This term, B(z), is here compared numerically to the second order term, A(z), of w for three different waves. It is seen that A(z) is more than 1oo times bigger than B(z) for shallow water and deep water waves, and A(z) is 1o times bigger than B(z) for transitional waves.

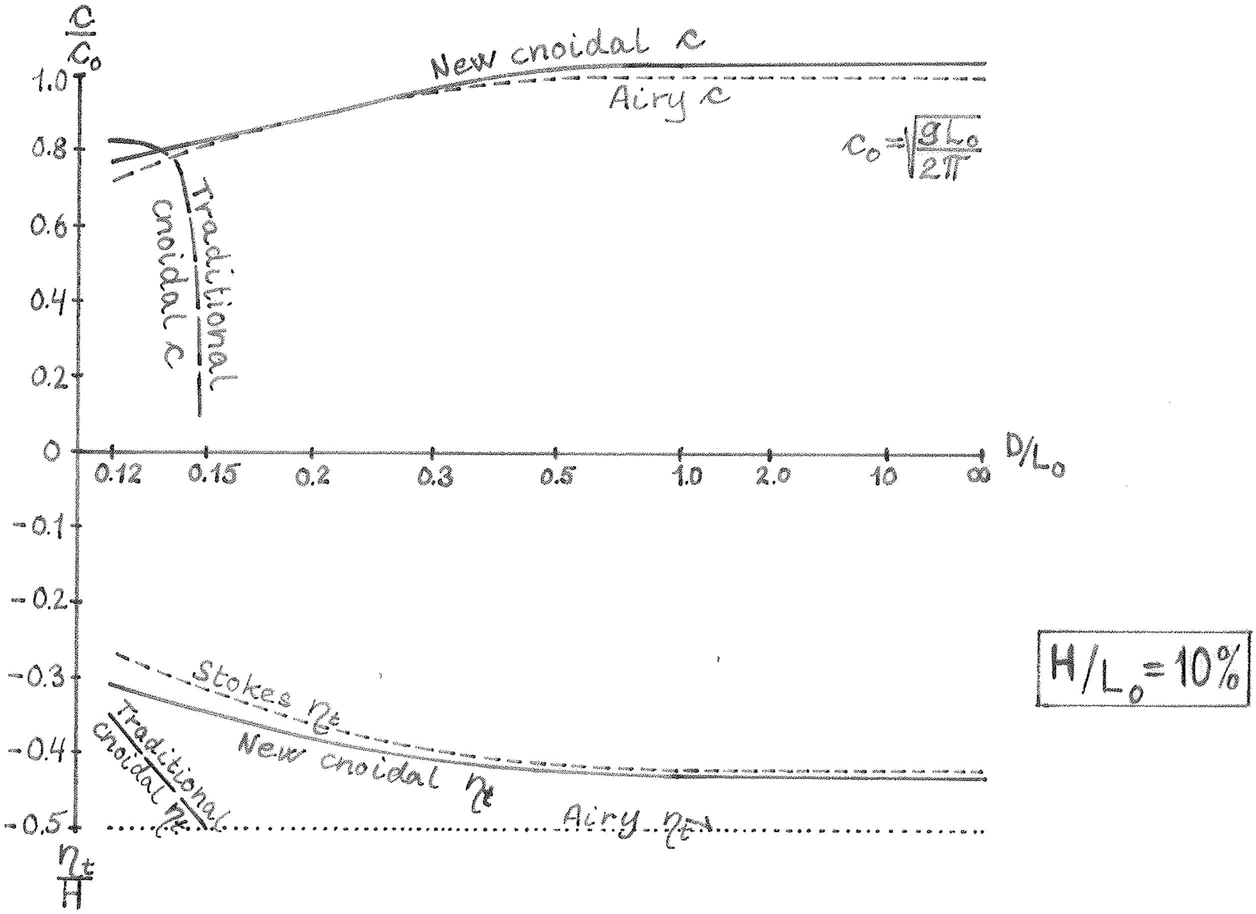


Fig. 14.

PRESSURE

The expressions for velocity and pressure will be considered again.

For u we have eq. 7

$$u = c\,\eta\, R\frac{\cosh Rz}{\sinh Ry} \tag{93}$$

which can be used for any type of wave, from deep water waves to solitary waves.

Below the crest of a solitary wave we find for R from eqs. 65 and 60

$$R = 1.2\ r = 1.2\ \frac{H}{D}\sqrt{\frac{H}{D}} \tag{94}$$

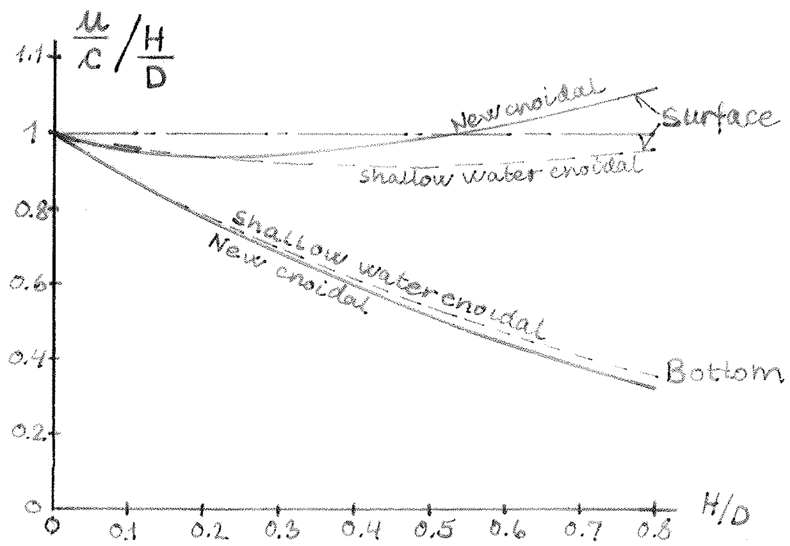


Fig. 15. The horizontal particle velocity at the surface and at the bottom of the crest of the solitary wave according to eq. 93 of this chapter compared to the shallow water expression of chapter VIII.

It can then be of interest to compare eq. 93 for the solitary wave with the equivalent expression for the solitary wave from chapter VIII. This is done in fig. 15. We see a small deviation for bigger H/D, which may indicate that r in eq. 94 is a little too big, although figs. 8 and 9 do not give this result.

In the third order cnoidal theory, terms with m^2 will be included in the expression for r, which is of importance for the solitary wave limit.

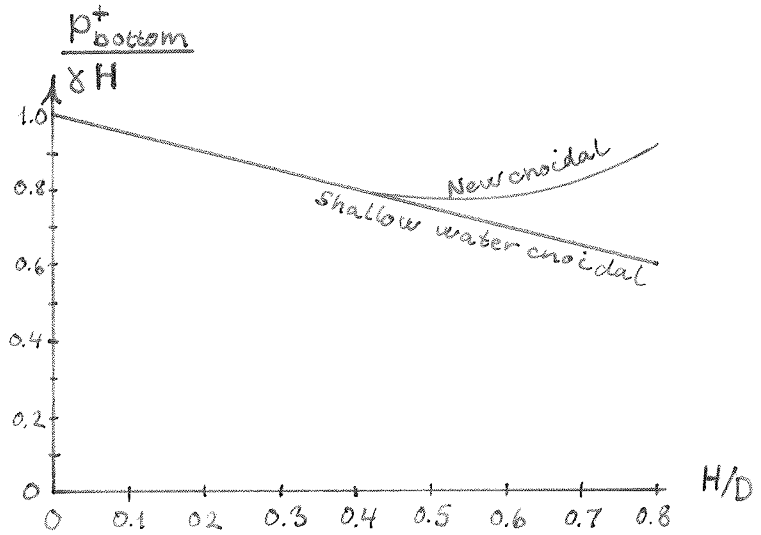


Fig. 16. The pressure at the bottom of the crest of the solitary wave according to eq. 95 of this chapter compared to the shallow water expression of chapter VIII.

For the pressure we use eq. 2o, which below the crest will give, when also the neglected terms are included,

$$\frac{p}{\gamma} = y - z + \frac{c^2}{gR}\frac{\partial^2\eta}{\partial x^2}\left\{[1 - R\eta\coth Ry]\,\frac{\cosh Ry - \cosh Rz}{\sinh Ry}\right.$$

$$\left. - \frac{1}{4}R\eta[1 - R\eta\coth Ry]\,\frac{\cosh 2Ry - \cosh 2Rz}{\sinh^2 Ry}\right\} \qquad (95)$$

Eq. 95 is shown in fig. 16 for the solitary wave, together with the result of chapter VIII. So the only neglected term for this situation was a third order term. The problem is that with R in eq. 94 the negligible third order term will be as big as the second order term for H/D near o.8. This means that we can as well propose the whole term with $\cosh 2Rz$ to be neglected, because it is without importance for higher waves and because it is small of second order for smaller waves. This gives us a more simple expression

$$\frac{p}{\gamma} = y - z + \frac{c^2}{gR}\frac{\partial^2\eta}{\partial x^2}[1 - R\eta\coth Ry]\,\frac{\cosh Ry - \cosh Rz}{\sinh Ry} \qquad (96)$$

but it is not a complete second order expression. For waves with $H/D > o.5$ the second order expression eq. 2o should not be used because the higher order terms are too important (unless a different expression for R is used) but instead eq. 96 can be used.

DIFFERENT CNOIDAL SURFACE PROFILE

It will here be shown how to develop the wave theory so that the shallow water limit will be exactly the traditional cnoidal wave.

We go back to the z-dependent part of the wave equation, to eq. 29. With eqs. 27 and 28 we got eq. 3o. If we had not substituted $\partial^2\eta/\partial x^2$ with eq. 27 we would have got

$$\frac{\partial^3\eta}{\partial x^3} = -\frac{\partial\eta}{\partial x}\left[r^2 - 6\,\frac{\partial^2\eta}{\partial x^2}\,k \coth kD + 4\beta\eta k^3 \right] \qquad (97)$$

With η from eq. 31 we get (see eq. 18 in chapter VIII)

$$\frac{\partial^2\eta}{\partial x^2} = -\frac{8K^2H}{L^2}\left[m - 1 - 2\,(2m - 1)\,cn^2\theta + 3m\,cn^4\theta \right] \qquad (98)$$

This is used in a second order term in eq. 97 so we use the sinusoidal approximation which we get for $m \to o$. Then eq. 98 will reduce to

$$\frac{\partial^2\eta}{\partial x^2} = \frac{8K^2H}{L^2}\left[1 - 2\,cn^2\theta \right] \qquad (99)$$

K is expanded in the Maclaurin series (see eq. 56 in chapter VIII)

$$K^2 = \frac{\pi^2}{4}\left(1 + \frac{m}{2} + \dots \right) \qquad (1oo)$$

Eqs. 99 and 1oo are used in eq. 97 together with eq. 36 to give

$$\frac{16\,K^2}{L^2}\left[1 - 2m + 3m\,cn^2\theta \right]$$

$$= r^2 - 24\,\frac{\pi^3}{L^2}\,\frac{H}{L}\,(1 + \frac{m}{2})(1 - 2\,cn^2\theta)\coth kD + 4\beta k^3 H\,cn^2\theta \qquad (1o1)$$

The $cn^2\theta$ -dependent equation then gives

$$mK^2 = \pi^3\,\frac{H}{L}\left[(1 + \frac{m}{2})\coth kD + \frac{2}{3}\,\beta \right] \qquad (1o2)$$

In the same way the z-independent equation eq. 43 and eq. 44 can be arranged to give for β

$$\beta = \frac{3}{2}\,\frac{\coth kD}{\sinh^2 kD}\,(1 + \frac{m}{2}) \qquad (1o3)$$

By this eq. 1o2 will be

$$mK^2 = [\ 1 + \frac{m}{2}\ ]\ \pi^3\ \frac{H}{L}\ \coth^3 kD \qquad (1o4)$$

For the shallow water limit we then find, for $m \rightarrow 1$,

$$mK^2 = \frac{3}{16}\ \frac{H}{D}\ (\frac{L}{D})^2 \qquad (1o5)$$

This is the same as for the traditional shallow water waves of Korteweg and de Vries.

But when substituting K in eq. 99 with eq. 1oo we can say that we in eq. 98 took one term along with m but dropped the other terms with m . This is correct as long as we do not claim eq. 1o4 to be determined with bigger accuracy. The third order theory will tell us if we should use [1 + m/2] on the right side of eq. 1o4, or we can use some special condition, like a demand to the solitary wave limit.

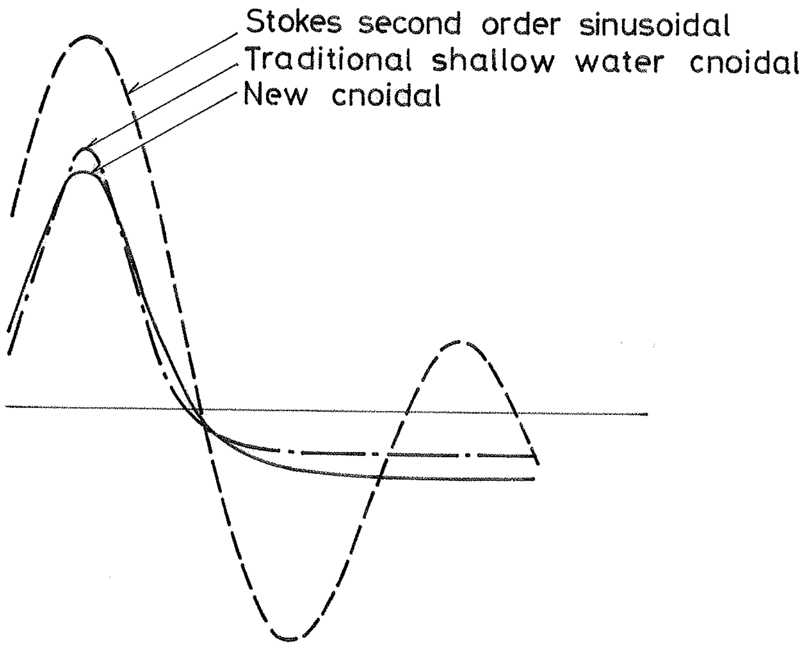


Fig.17. Using the equations tabulated in chapter X we find the shallow water limit of the new cnoidal wave slightly different from the traditional cnoidal wave, but in this appendix it is shown how to express the new wave so it will coincide with the traditional wave.