CHAPTER VIII

PROGRESSIVE CNOIDAL SHALLOW WATER WAVES

ABSTRACTS

In this chapter the traditional shallow water cnoidal waves are considered. It is shown how they can be developed in somewhat the same way as the waves of the preceding chapters, avoiding potential functions. In this way the wave does not need to be irrotational. Otherwise the wave profile will be the same as the traditional one, given by e.g. Keulegan and Patterson in 194o. Most of this chapter is a copy of a thesis by the author from 1971, with the expressions for particle velocities, pressure and energies, and with graphs for calculations of cnoidal waves and the solitary wave.

INTRODUCTION

In the preceding chapters we have developed different sinusoidal waves. In the first order wave theory we found that we had to neglect terms that were not so negligible for realistic waves. For the second order wave we found that the sinusoidal theory could lead to unrealistic results specially for waves on shallow waters. So we turn our attention to the shallow water cnoidal waves.

In chapter VI we developed the cnoidal wave on infinite deep water. This was not so very complicated. The cnoidal wave on shallow waters may seem to be a little more complicated. But they have been known since 1895 (Korteweg and de Vries). Reasonable expressions for the particle velocities, the pressure, and the energies have though been lacking. Those copied here were found in good agreement with experiments.

In the next chapter a different theory will be presented that includes also the shallow water, a theory that does not neglect some questionable terms.

BASIC EQUATIONS

In the first order sinusoidal theory, chapter IV, we found the expressions for u, w, and p

$$u = c \eta k \frac{\cosh kz}{\sinh ky} \tag{1}$$

$$w = - c \frac{\partial \eta}{\partial x} \frac{\sinh kz}{\sinh ky} \tag{2}$$

$$\frac{p}{\gamma} = D + \eta - z + \frac{1}{g} \frac{\partial^2 \eta}{\partial t^2} \frac{1}{k} \frac{\cosh ky - \cosh kz}{\sinh ky} \tag{3}$$

Expanding cosh and sinh in Maclaurin series we get

$$\cosh kz = 1 + \frac{(kz)^2}{2} + \dots \tag{4}$$

$$\sinh ky = ky + \frac{(ky)^3}{6} + \dots \tag{5}$$

In a first order theory we can substitute $y = D + \eta$ by D. We then get for the shallow water limit of eqs. 1,2, and 3, i.e. for $D/L \to o$, the well known

$$u = c \frac{\eta}{D} \tag{6}$$

$$w = - c \frac{\partial \eta}{\partial x} \frac{z}{D} \tag{7}$$

$$\frac{p}{\gamma} = D + \eta - z \tag{8}$$

But including still a term in eq. 4 we see that u also depends on z^2, so we write for u

$$u = u_b + u_{b1} z + u_{b2} z^2 + c \left(\frac{H}{D}\right)^2 F(z) \tag{9}$$

$u_b = u_b(x,t)$ is the velocity at the bottom $z = o$. u_{b1} and u_{b2} are arbitrary functions of x and t. $F(z)$ is an arbitrary function chosen so that the last term has a second order magnitude.

For F(z) we want the condition to be imposed

$$\int_o^y F(z)\, dz = o \qquad (10)$$

and for u we remember the condition

$$q = c\,\eta = \int_o^y u\, dz \qquad (11)$$

The equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = o \qquad (12)$$

and w = o at the bottom gives for w using eq. 9

$$w = -\frac{\partial u_b}{\partial x} z - \frac{1}{2}\frac{\partial u_{b1}}{\partial x} z^2 - \frac{1}{3}\frac{\partial u_{b2}}{\partial x} z^3 \qquad (13)$$

The rotation will be

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = c\,(\frac{H}{2})^2 \frac{\partial F(z)}{\partial z} + \frac{1}{3}\frac{\partial^2 u_{b2}}{\partial x^2} z^3$$

$$+ \; u_{b1} + (2\,u_{b2} + \frac{\partial^2 u_b}{\partial x^2})\, z + \frac{1}{2}\frac{\partial^2 u_{b1}}{\partial x^2} z^2 \qquad (14)$$

Ω must be a constant in time, so we get

$$u_{b1} = o \qquad (15)$$

as could be expected from eq. 4, and

$$2u_{b2} = -\frac{\partial^2 u_b}{\partial x^2} \qquad (16)$$

Then eqs. 9, 13, and 14 are changed to

$$u = u_b - \frac{1}{2}\frac{\partial^2 u_b}{\partial x^2} z^2 + c\,(\frac{H}{D})^2 F(z) \qquad (17)$$

$$w = -\frac{\partial u_b}{\partial x} z + \frac{1}{6}\frac{\partial^3 u_b}{\partial x^3} z^3 \qquad (18)$$

$$\Omega = c\,(\frac{H}{D})^2 \frac{\partial F(z)}{\partial z} - \frac{1}{6}\frac{\partial^4 u_b}{\partial x^4} z^3 \qquad (19)$$

Later we can show that the last term in eq. 19 is negligible in this theory, so the rotation will have the second order value

$$\Omega = c \left(\frac{H}{D}\right)^2 \frac{\partial F(z)}{\partial z} \tag{2o}$$

So the arbitrary function $F(z)$ was included in eq. 9 simply to get a desired rotation. Choosing $\partial F(z)/\partial z = o$ the wave will be irrotational, like the classical cnoidal waves.

We will later find that $\frac{\partial^2 u_b}{\partial x^2}$ and $\frac{\partial^3 u_b}{\partial x^3}$ are of second and third order in this shallow water theory. This means that they can be substituted by the simple first order shallow water expression of eq. 6. Then w, eq. 18, will be

$$w = -\frac{\partial u_b}{\partial x} z + \frac{1}{6} c \frac{1}{y} \frac{\partial^3 \eta}{\partial x^3} z^3 \tag{21}$$

At the surface $z = y = D + \eta$ this will be, using eq. 6

$$w_s = -\frac{\partial u_b}{\partial x} D - c \frac{\partial \eta}{\partial x} \frac{\eta}{D} + \frac{1}{6} c \frac{\partial^3 \eta}{\partial x^3} D^2 \tag{22}$$

We here used that in second order terms y can be approximated by D when needed.

The kinematic surface condition

$$w_s = \frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} \tag{23}$$

then gives, using eq. 6 for u_s in the last, second order term,

$$-\frac{\partial u_b}{\partial x} D - c \frac{\partial \eta}{\partial x} \frac{\eta}{D} + \frac{1}{6} c \frac{\partial^3 \eta}{\partial x^3} D^2 = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} c \frac{\eta}{D} \tag{24}$$

or

$$-\frac{\partial u_b}{\partial x} D = \frac{\partial \eta}{\partial t} + c \frac{\partial}{\partial x} \left(\frac{\eta^2}{D} - \frac{1}{6} \frac{\partial^2 \eta}{\partial x^2} D^2\right) \tag{25}$$

For a permanent progressive wave we have

$$\frac{\partial \eta}{\partial t} = -c \frac{\partial \eta}{\partial x} \tag{26}$$

and in second order terms we can use the sinusoidal first order expression for c

$$c = \sqrt{g D} \tag{27}$$

Then eq. 25 will be after differentation with respect to t

$$- \frac{\partial^2 u_b}{\partial x \partial t} D = \frac{\partial^2 \eta}{\partial t^2} - g D \frac{\partial^2}{\partial x^2} (\frac{\eta^2}{D} - \frac{D^2}{6} \frac{\partial^2 \eta}{\partial x^2}) \tag{28}$$

The horizontal dynamic equation

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \tag{29}$$

is wanted at the surface, $z = y$.

In second order terms we use $y = D$ and the sinusoidal expressions, eqs. 6 and 7 so with eqs. 17 and 26 we get

$$G_{xs} = \frac{\partial u_b}{\partial t} + c^2 \frac{D}{2} \frac{\partial^3 \eta}{\partial x^3} + c^2 \frac{1}{2D^2} \frac{\partial(\eta^2)}{\partial x} \tag{3o}$$

For sinusoidal waves the pressure is hydrostatic as given by eq. 8. For the classical shallow water cnoidal waves the pressure is also assumed to be hydrostatic, so

$$\frac{1}{\gamma} \frac{\partial p}{\partial x} = \frac{\partial \eta}{\partial x} \tag{31}$$

Comparing with eq. 3 we see that this means that the influence of the vertical acceleration is neglected. For infinite small cnoidal waves this may seem reasonable, but not for more realistic waves. In chapter IX we find a different cnoidal wave for arbitrary depth. Then we include the vertical acceleration term in the pressure.

The horizontal equation of momentum

$$- \frac{\partial p}{\partial x} = \rho G_x \tag{32}$$

then gives for the surface with eqs. 3o and 31, and substituting c in second order terms with eq. 27

$$- \frac{\partial \eta}{\partial x} = \frac{1}{g} [\frac{\partial u_b}{\partial t} + g \frac{D^2}{2} \frac{\partial^3 \eta}{\partial x^3} + g \frac{1}{2D} \frac{\partial(\eta^2)}{\partial x}] \tag{33}$$

which is differentiated with respect to x

$$- \frac{\partial^2 u_b}{\partial x \partial t} = g \frac{\partial^2 \eta}{\partial x^2} + g \cdot \frac{\partial^2}{\partial x^2} (\frac{\eta^2}{2 D} + \frac{D^2}{2} \frac{\partial^2 \eta}{\partial x^2}) \tag{34}$$

Together with eq. (28) this gives

$$\frac{\partial^2\eta}{\partial t^2} = g\ D\ \frac{\partial^2\eta}{\partial x^2} + g\ D\ \frac{\partial^2}{\partial x^2}\left(\frac{3}{2}\ \frac{\eta^2}{D} + \frac{D^2}{3}\ \frac{\partial^2\eta}{\partial x^2}\right) \tag{35}$$

which is the governing differential equation for the shallow water cnoidal wave as given by Keulegan and Patterson. Their procedure has been followed to some extent, and their theory can be used from here on to give the expressions for the wave parameters : the celerity C, the wave length L, the crest height η_c, and the surface elevation η.

The following is a copy of the author's thesis from 1971. The notation used can be found at the end of the chapter. It is different from the rest of this thesis at only two points : instead of the parameter m we here use the modulus k in the Jacobian elliptic functions, where we have $m = k^2$ (the modulus is used in the courses of mathematics at the Technical University of Denmark). In the part with transition from cnoidal to sinusoidal theory $2\pi/L$ is then called R instead of k, as we for the first order theory of chapter IV get $R = 2\pi/L$. The symbol y is not used for the actual water depth $D + \eta$, which makes the expressions look longer.

The equations from 1971 are indicated with ('), e.g. the celerity is given in (7'). Eqs. (1') to (6') and (14') to (16') are not taken along here.

The celerity c or C will be

$$\frac{C^2}{g \cdot D} = 1 + \frac{H}{D \cdot k^2} \cdot \left(2 - k^2 - 3 \cdot \frac{E(k)}{K(k)}\right) \tag{7'}$$

The wave-length, L, is:

$$\frac{L}{D} = \frac{4}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot k \cdot K(k) \qquad (8')$$

From (7') and (8') the wave-period, T, will be:

$$T^2 = \frac{L^2}{C^2} = \frac{\frac{16}{3} \cdot D^2 \cdot K^2(k) \cdot k^2}{g \cdot H \cdot \left[1 + \frac{H}{D \cdot k^2} \cdot \left(2 - k^2 - 3 \cdot \frac{E(k)}{K(k)}\right)\right]} \qquad (9')$$

The crest-height, η_c, will be:

$$\eta_c = \frac{H}{k^2} \cdot \left(1 - \frac{E(k)}{K(k)}\right) \qquad (10')$$

and thereby the trough-depth, η_t, (negative):

$$\eta_t = \eta_c - H = \frac{H}{k^2} \cdot \left(1 - k^2 - \frac{E(k)}{K(k)}\right) \qquad (11')$$

To avoid misunderstandings the absolute value $|\eta_t|$ will be used hereafter. With cn being the Jacobian elliptic cosine function the free surface profile, η, will be:

$$\eta = - |\eta_t| + H \cdot cn^2 \theta \qquad (12')$$

where θ is equal to:

$$\theta = 2 \cdot K(k) \cdot \left(\frac{x}{L} - \frac{t}{T}\right) \qquad (13')$$

in which x is the horizontal co-ordinate and t the time.

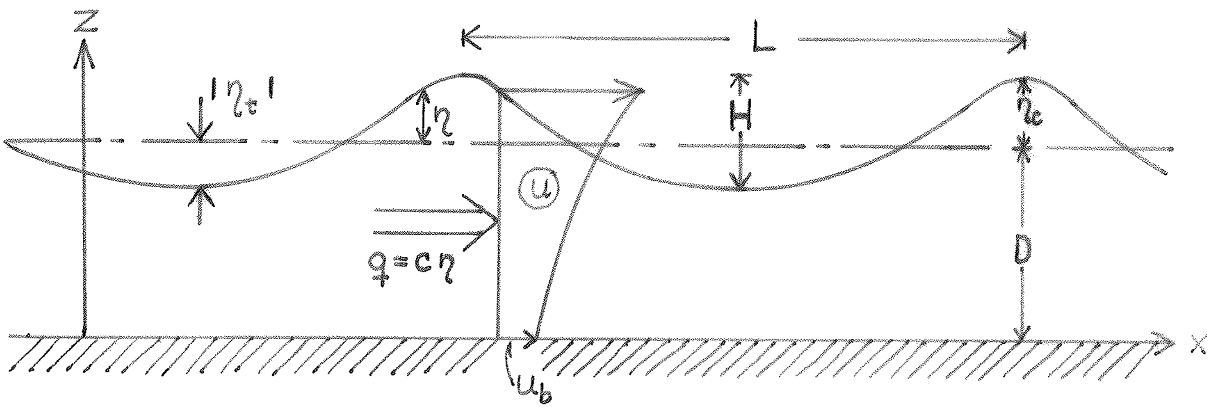


Fig. 1. In the shallow water limit the sinusoidal theory will give a parabolic velocity distribution. This can be used in second order terms to find the cnoidal theory.

PARTICLE VELOCITIES

The derivatives of η are got from (12') and (13'):

$$\frac{\partial \eta}{\partial x} = - \frac{4 \cdot K(k) \cdot H}{L} \cdot \text{sn}\ \theta \cdot \text{cn}\ \theta \cdot \text{dn}\ \theta = - \frac{4 \cdot K(k) \cdot H}{L} \cdot \sqrt{\text{cn}^2\theta \cdot (1 - \text{cn}^2\theta) \cdot (k_c^2 + k^2 \cdot \text{cn}^2\theta)} \tag{17'}$$

$$\frac{\partial^2 \eta}{\partial x^2} = - \frac{8 \cdot K^2(k) \cdot H}{L^2} \cdot (\text{cn}^2\theta \cdot \text{dn}^2\theta - \text{sn}^2\theta \cdot \text{dn}^2\theta - k^2 \cdot \text{sn}^2\theta \cdot \text{cn}^2\theta) = - \frac{8 \cdot K^2(k) \cdot H}{L^2} \cdot \left[- k_c^2 - 2 \cdot (k^2 - k_c^2) \cdot \text{cn}^2\theta + 3 \cdot k^2 \cdot \text{cn}^4\theta\right] \tag{18'}$$

$$\frac{\partial^3 \eta}{\partial x^3} = \frac{64 \cdot K^3(k) \cdot H}{L^3} \cdot (-k^2 \cdot sn^2\theta + k^2 \cdot cn^2\theta + dn^2\theta)$$

$$\cdot\ sn\ \theta \cdot cn\ \theta \cdot dn\ \theta = \frac{64 \cdot K^3(k) \cdot H}{L^3} \cdot (k_c^2 - k^2 + 3 \cdot k^2$$

$$\cdot\ cn^2\theta) \cdot \sqrt{cn^2\theta \cdot (1 - cn^2\theta) \cdot (k_c^2 + k^2 \cdot cn^2\theta)} \qquad (19')$$

The horizontal velocity is found from eq. 17. We will consider the situation with $F(z) = o$, i.e. for irrotational motion. Then we get

$$u(z) = u_b - \frac{z^2}{2} \cdot \frac{\partial^2 u_b}{\partial x^2} \qquad (20')$$

The water discharge, Q, through a vertical is then:

$$Q = \int_o^{D + \eta} u(z) \cdot dz = u_b \cdot (D + \eta) - \frac{(D + \eta)^3}{6} \cdot \frac{\partial^2 u_b}{\partial x^2} \qquad (21')$$

By looking at two closely spaced verticals it can be deduced that for permanent waves in water with no stream, Q will also be:

$$Q = C \cdot \eta \qquad (22')$$

Determining μ later the following substitution is made:

$$(D + \eta) \cdot \mu \cdot C \cdot \frac{\partial^2 \eta}{\partial x^2} = \frac{(D + \eta)^3}{6} \cdot \frac{\partial^2 u_b}{\partial x^2} \qquad (23')$$

(21'), (22') and (23') are then combined:

$$C \cdot \eta = u_b \cdot (D + \eta) - (D + \eta) \cdot \mu \cdot C \cdot \frac{\partial^2 \eta}{\partial x^2}$$

$$u_b = C \cdot \frac{\eta}{D + \eta} + \mu \cdot C \cdot \frac{\partial^2 \eta}{\partial x^2} \qquad (24')$$

If as an approximation sinusoidal wave expressions are used with $(D + \eta)^3 \simeq (D + \eta) \cdot D^2$ and $u_b = \frac{C}{D} \cdot \eta$ (23') yields:

$$\mu = \frac{D}{6} \qquad (25')$$

So the derivatives of μ in smaller terms are put to 0, whereby (24') with terms of no more than 3rd order in derivation or power or a combination of derivation and power yields:

$$\frac{\partial u_b}{\partial x} = C \cdot \frac{D}{(D + \eta)^2} \cdot \frac{\partial \eta}{\partial x} + \mu \cdot C \cdot \frac{\partial^3 \eta}{\partial x^3} \qquad (26')$$

$$\frac{\partial^2 u_b}{\partial x^2} = C \cdot D \cdot \left[- \frac{2}{(D + \eta)^3} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{1}{(D + \eta)^2} \cdot \frac{\partial^2 \eta}{\partial x^2}\right] \qquad (27')$$

$$\frac{\partial^3 u_b}{\partial x^3} = C \cdot D \cdot \left[\frac{6}{(D + \eta)^4} \cdot \left(\frac{\partial \eta}{\partial x}\right)^3 - \frac{6}{(D + \eta)^3} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2}\right.$$

$$\left. + \frac{1}{(D + \eta)^2} \cdot \frac{\partial^3 \eta}{\partial x^3}\right] \qquad (28')$$

μ is found from (21'), (22'), (24') and (27') whereby (20') gives u(z):

$$u(z) = C \cdot \left\{ \frac{\eta}{D + \eta} + \frac{3 \cdot z^2 - (D + \eta)^2}{6 \cdot (D + \eta)^2} \cdot D \cdot \left[\frac{2}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 - \frac{\partial^2 \eta}{\partial x^2} \right] \right\} \qquad (29')$$

The vertical particle velocity, w(z) , is got from eq. 18

$$w(z) = - z \cdot \frac{\partial u_b}{\partial x} + \frac{z^3}{6} \cdot \frac{\partial^3 u_b}{\partial x^3}$$

Here sufficient accuracy is obtained by using (25') in (26') whereby also (28') gives:

$$w(z) = C \cdot \left\{ - z \cdot \left[\frac{D}{(D + \eta)^2} \cdot \frac{\partial \eta}{\partial x} + \frac{D}{6} \cdot \frac{\partial^3 \eta}{\partial x^3} \right] + z^3 \cdot \frac{D}{(D + \eta)^2} \cdot \left[\frac{1}{(D + \eta)^2} \cdot \left(\frac{\partial \eta}{\partial x}\right)^3 - \frac{1}{D + \eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{6} \cdot \frac{\partial^3 \eta}{\partial x^3} \right] \right\} \qquad (30')$$

Instead of using (25') it is possible in (26') to include the term with $\partial\mu/\partial x$ by using (23') and (27'). In (26') the term $\mu C \frac{\partial^3 \eta}{\partial x^3}$ wil then be changed with $C\frac{D}{6}\left[\frac{\partial^3 \eta}{\partial x^3} + \frac{2}{(D+\eta)^2}\left(\frac{\partial \eta}{\partial x}\right)^3 - \frac{4}{D+\eta}\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right]$
This will make a slight change in the expression for w(z) in (3o').

PARTICLE ACCELERATIONS

By differentiating $u(z)$, (29') and $w(z)$, (30') we get

$$\frac{\partial u(z)}{\partial t} = C \cdot \Big\{\frac{D}{(D+\eta)^2} \cdot \frac{\partial \eta}{\partial t} - \frac{3 \cdot z^2 - (D+\eta)^2}{6 \cdot (D+\eta)^2} \cdot D$$

$$\cdot \Big[\frac{6}{(D+\eta)^2} \cdot \frac{\partial \eta}{\partial t} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^2 - \frac{4}{D+\eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}$$

$$- \frac{2}{D+\eta} \cdot \frac{\partial \eta}{\partial t} \cdot \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^3 \eta}{\partial x^2 \cdot \partial t}\Big] - \frac{D}{3 \cdot (D+\eta)}$$

$$\cdot \Big[\frac{2}{D+\eta} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^2 - \frac{\partial^2 \eta}{\partial x^2}\Big] \cdot \frac{\partial \eta}{\partial t}\Big\} \qquad (31')$$

$$\frac{\partial u(z)}{\partial x} = C \cdot \Big\{\frac{D}{(D+\eta)^2} \cdot \frac{\partial \eta}{\partial x} - \frac{3 \cdot z^2 - (D+\eta)^2}{6 \cdot (D+\eta)^2} \cdot D \cdot$$

$$\Big[\frac{6}{(D+\eta)^2} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^3 - \frac{6}{D+\eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^3 \eta}{\partial x^3}\Big]$$

$$- \frac{D}{3 \cdot (D+\eta)} \cdot \Big[\frac{2}{D+\eta} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^2 - \frac{\partial^2 \eta}{\partial x^2}\Big] \cdot \frac{\partial \eta}{\partial x}\Big\} \qquad (32')$$

$$\frac{\partial u(z)}{\partial z} = C \cdot z \cdot \frac{D}{(D+\eta)^2} \cdot \Big[\frac{2}{D+\eta} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^2 - \frac{\partial^2 \eta}{\partial x^2}\Big] \qquad (33')$$

$$\frac{\partial w(z)}{\partial t} = C \cdot \Big\{- z \cdot \Big[- \frac{2 \cdot D}{(D+\eta)^3} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial t} + \frac{D}{(D+\eta)^2} \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t}$$

$$+ \frac{D}{6} \cdot \frac{\partial^4 \eta}{\partial x^3 \cdot \partial t}\Big] + z^3 \cdot \frac{D}{(D+\eta)^2} \cdot \Big[- \frac{4}{(D+\eta)^3} \cdot \frac{\partial \eta}{\partial t} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^3$$

$$+ \frac{3}{(D+\eta)^2} \cdot \Big(\frac{\partial \eta}{\partial x}\Big)^2 \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t} + \frac{3}{(D+\eta)^2} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial t} \cdot \frac{\partial^2 \eta}{\partial x^2}$$

$$- \frac{1}{D+\eta} \cdot \frac{\partial^2 \eta}{\partial x \cdot \partial t} \cdot \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{D+\eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^3 \eta}{\partial x^2 \cdot \partial t}$$

$$- \frac{1}{3 \cdot (D+\eta)} \cdot \frac{\partial \eta}{\partial t} \cdot \frac{\partial^3 \eta}{\partial x^3} + \frac{1}{6} \cdot \frac{\partial^4 \eta}{\partial x^3 \cdot \partial t}\Big]\Big\} \qquad (34')$$

$$\frac{\partial w(z)}{\partial x} = C \cdot \Big\{- z \cdot \Big[- \frac{2 \cdot D}{(D+\eta)^3} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{D}{(D+\eta)^2} \cdot \frac{\partial^2 \eta}{\partial x^2}$$

$$+ \frac{D}{6} \cdot \frac{\partial^4 \eta}{\partial x^4}\Big] + z^3 \cdot \frac{D}{(D+\eta)^2} \cdot \Big[- \frac{4}{(D+\eta)^3} \cdot \left(\frac{\partial \eta}{\partial x}\right)^4 + \frac{6}{(D+\eta)^2}$$

$$\cdot \left(\frac{\partial \eta}{\partial x}\right)^2 \cdot \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{D+\eta} \cdot \left(\frac{\partial^2 \eta}{\partial x^2}\right)^2 - \frac{1}{D+\eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^3 \eta}{\partial x^3}$$

$$- \frac{1}{3 \cdot (D+\eta)} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^3 \eta}{\partial x^3} + \frac{1}{6} \cdot \frac{\partial^4 \eta}{\partial x^4}\Big]\Big\} \qquad (35')$$

$$\frac{\partial w(z)}{\partial z} = C \cdot \Big\{- \frac{D}{(D+\eta)^2} \cdot \frac{\partial \eta}{\partial x} - \frac{D}{6} \cdot \frac{\partial^3 \eta}{\partial x^3} + z^2 \cdot \frac{D}{(D+\eta)^2}$$

$$\cdot \Big[\frac{3}{(D+\eta)^2} \cdot \left(\frac{\partial \eta}{\partial x}\right)^3 - \frac{3}{D+\eta} \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2} + \frac{1}{2} \cdot \frac{\partial^3 \eta}{\partial x^3}\Big]\Big\} \qquad (36')$$

The horizontal particle acceleration is then

$$\frac{du(z)}{dt} = \frac{\partial u(z)}{\partial t} + u(z) \cdot \frac{\partial u(z)}{\partial x} + w(z) \cdot \frac{\partial u(z)}{\partial z} \qquad (37')$$

and the vertical will be written as:

$$\frac{dw(z)}{dt} = \frac{\partial w(z)}{\partial t} + u(z) \cdot \frac{\partial w(z)}{\partial x} + w(z) \cdot \frac{\partial w(z)}{\partial z}$$

$$= B_1(\eta) \cdot Z + B_2(\eta) \cdot Z^3 + B_3(\eta) \cdot Z^5 \qquad (38')$$

where $B_1(\eta)$, $B_2(\eta)$ and $B_3(\eta)$ are found from (34'), (29'), (35'), (30') and (36'). In many cases it will be reasonable to include no more than 3^{rd} order terms whereby $B_2(\eta) = 0$ and $B_3(\eta) = 0$. Using $\frac{\partial \eta}{\partial t} = - C \cdot \frac{\partial \eta}{\partial x}$ and

$\frac{\partial^2 \eta}{\partial x \cdot \partial t} = - C \cdot \frac{\partial^2 \eta}{\partial x^2}$ $B_1(\eta)$ will be:

$$B_1(\eta) = \frac{C^2 \cdot D^2}{(D + \eta)^3} \cdot \left[- \frac{1}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2 \eta}{\partial x^2}\right] \qquad (39')$$

PRESSURE

The vertical dynamic equation of an infinitesimal unit cube yields:

$$\frac{\partial p(z)}{\partial z} = - \gamma - \frac{\gamma}{g} \cdot \frac{dw(z)}{dt}$$

where γ is the unit weight of the water. The pressure in the water, $p(z)$, above atmospheric pressure is found by integrating from the free surface $D + \eta$ to z, using (38') and (39'):

$$p(z) = \gamma \cdot (D + \eta - z) + \frac{\gamma}{2g} \cdot \frac{C^2 \cdot D^2}{(D + \eta)^3} \cdot \left[- \frac{1}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2 \eta}{\partial x^2}\right] \cdot \left[(D + \eta)^2 - z^2\right] \qquad (40')$$

The wave-pressure, $p^+(z)$, for $z \leq D$ is then:

$$p^+(z) = p(z) - \gamma \cdot (D - z)$$

POINT OF INFLECTION AND MEAN WATER LEVEL

The complementary modulus, k_c is defined as:

$$k_c = \sqrt{1 - k^2}$$

The point of inflection is got from (18') for

$$\frac{\partial^2 \eta}{\partial x^2} = 0 \text{ , with } \theta = \theta_v :$$

$$3 \cdot k^2 \cdot cn^4\theta_v - 2 \cdot (k^2 - k_c^2) \cdot cn^2\theta_v - k_c^2 = 0$$

$$cn^2\theta_v = \frac{k^2 - k_c^2}{3k^2} \overset{+}{(-)} \sqrt{\left(\frac{k^2 - k_c^2}{3 \cdot k^2}\right)^2 + \frac{k_c^2}{3 \cdot k^2}} \qquad (41')$$

where the negative sign has to be omitted to get a positive $cn^2\theta$. From the definition of the incomplete elliptic integral of the first kind, $F(\phi,k)$, it is got, with Θ denoting a variable:

$$F(\phi,k) = \int_0^{\phi} \frac{d\Theta}{\sqrt{1 - k^2 \cdot \sin^2\Theta}} = \theta \qquad (42')$$

The Jacobian elliptic cosine function, cn is then defined as:

$$cn\ \theta = \cos\ \phi$$

From (41') a ϕ_v is thereby got by:

$$\phi_v = \text{Arccos}\ (\sqrt{cn^2\theta_v})$$

and then by (42'):

$$\theta_v = F(\phi_v, k)$$

With $x = 0$ at the crest the co-ordinates of the point of inflection, (λ_v, η_v), are then got from (13'), (12') and (41'):

$$(\lambda_v , \eta_v) = \left(\frac{L}{2 \cdot K(k)} \cdot \theta_v , - |\eta_t| + H \cdot cn^2\theta_v\right) \tag{43'}$$

The point of mean water level is found in a similar manner. In (12') $\eta = 0$ for $\theta = \theta_o$, so:

$$cn^2\theta_o = \frac{|\eta_t|}{H}$$

$$\phi_o = \text{Arccos } (\sqrt{cn^2\theta_o})$$

and then:

$$\theta_o = F (\phi_o , k) \tag{44'}$$

Like above the co-ordinates will be:

$$(\lambda_o , \eta_o) = \left(\frac{L}{2 \cdot K(k)} \cdot \theta_o , 0\right)$$

$$= \left(\frac{L}{2 \cdot K(k)} \cdot F\left[\text{Arccos } \sqrt{\frac{|\eta_t|}{H}} , k\right], 0\right) \tag{45'}$$

From Byrd and Friedman, reference [3], we get

$$\int cn^4\theta \cdot d\theta = \frac{1}{3 \cdot k^4} \cdot \Big[(2 - 3 \cdot k^2) \cdot k_c^2 \cdot \theta + 2$$

$$\cdot (2 \cdot k^2 - 1) \cdot E(\phi , k) + k^2 \cdot sn\ \theta \cdot cn\ \theta \cdot dn\ \theta\Big]$$

where, with Θ denoting a variable:

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2\Theta}\ d\Theta$$

is the incomplete elliptic integral of the second kind, with ϕ here chosen so that $F(\phi, k) = \theta$.

$\phi = \frac{\pi}{2}$ is equivalent to $\theta = F(\phi, k) = K(k)$ and

$E(\phi, k) = E(k)$. Hereby it is got:

$$\int_0^{K(k)} cn^4\theta \cdot d\theta = \frac{1}{3 \cdot k^4} \cdot \Big[(2 - 3 \cdot k^2) \cdot k_c^2 \cdot K(k)$$

$$+ 2 \cdot (2 \cdot k^2 - 1) \cdot E(k)\Big] \qquad (46')$$

With the same symbols there is got:

$$E(\phi, k) = k_c^2 \cdot \theta + k^2 \cdot \int_0^{\theta} cn^2\Theta \cdot d\Theta \qquad (47')$$

and for $\theta = K(k)$ above, changing Θ to θ below:

$$\int_0^{K(k)} cn^2\theta \cdot d\theta = \frac{1}{k^2} \cdot (E(k) - k_c^2 \cdot K(k)) \qquad (48')$$

The mean value of the potential energy per horizontal unit area, E_{pot}, is, using (12'), (46') and (48'):

$$E_{pot} = \frac{1}{L/2} \cdot \int_0^{L/2} \frac{\gamma}{2} \cdot \eta^2 \cdot dx = \frac{\gamma}{L} \cdot \int_0^{L/2} \eta_t^2 \cdot dx$$

$$- \frac{\gamma}{L} \cdot 2 \cdot |\eta_t| \cdot \int_0^{L/2} H \cdot cn^2\theta \cdot dx + \frac{\gamma}{L} \cdot \int_0^{L/2} H^2 \cdot cn^4\theta \cdot dx$$

$$= \frac{\gamma}{2} \cdot \eta_t^2 - \gamma \cdot \frac{H}{k^2 \cdot K(k)} \cdot |\eta_t| \cdot \left[E(k) - k_c^2 \cdot K(k)\right]$$

$$+ \frac{\gamma}{6} \cdot \frac{H^2}{k^4 \cdot K(k)} \cdot \left[(2 - 3 \cdot k^2) \cdot k_c^2 \cdot K(k)\right.$$

$$\left. + 2 \cdot (2 \cdot k^2 - 1) \cdot E(k)\right] \qquad (49')$$

Disregarding terms of 4^{th} and higher order (29') and (30') give:

$$\frac{u^2(z) + w^2(z)}{C^2} = \frac{\eta^2}{(D + \eta)^2} - \frac{\eta \cdot D}{3 \cdot (D + \eta)}$$

$$\cdot \left[\frac{2}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 - \frac{\partial^2 \eta}{\partial x^2}\right] + \left\{\frac{\eta \cdot D}{(D + \eta)^3} \cdot \left[\frac{2}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 - \frac{\partial^2 \eta}{\partial x^2}\right]\right.$$

$$\left. + \frac{D^2}{(D + \eta)^4} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2\right\} \cdot z^2$$

For the kinetic energy per horizontal unit area in a vertical cut, $\Delta E_{kin}(\eta)$, it is then got:

$$\Delta E_{kin}(\eta) = \int_0^{D+\eta} \frac{1}{2} \cdot \frac{\gamma}{g} \cdot \left[u^2(z) + w^2(z)\right] \cdot dz$$

$$= \frac{\gamma \cdot C^2}{2 \cdot g} \cdot \left[\frac{\eta^2}{D+\eta} + \frac{1}{3} \cdot \frac{D^2}{D+\eta} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2\right] \qquad (50')$$

The mean value of the kinetic energy per horizontal unit area, E_{kin} is then:

$$E_{kin} = \frac{1}{L} \cdot \int_0^L \Delta E_{kin}(\eta) \cdot dx \qquad (51')$$

which is found by numerical integration.

Disregarding terms of 4^{th} and higher order, (29'), (40') and later (30') give:

$$\frac{[p(z,t) + \gamma \cdot z] \cdot u(z)}{\gamma \cdot C} = \eta - \frac{(D+\eta) \cdot D}{6}$$

$$\cdot \left[\frac{2}{D+\eta} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\right] + \frac{1}{2 \cdot g} \cdot \frac{\eta}{(D+\eta)^2} \cdot C^2 \cdot D^2 \cdot$$

$$\left[- \frac{1}{D+\eta} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial^2\eta}{\partial x^2}\right] + \left\{\frac{1}{2} \cdot \frac{D}{D+\eta}\right.$$

$$\cdot \left[\frac{2}{D+\eta} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\right] - \frac{1}{2 \cdot g} \cdot \frac{\eta}{(D+\eta)^4} \cdot C^2 \cdot D^2$$

$$\left.\cdot \left[- \frac{1}{D+\eta} \left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial^2\eta}{\partial x^2}\right]\right\} \cdot z^2$$

$$\frac{(u^2(z) + w^2(z)) \cdot u(z)}{C^3} = \frac{\eta^3}{(D+\eta)^3}$$

$$- \frac{1}{2} \cdot \frac{\eta^2 \cdot D}{(D+\eta)^2} \cdot \left[\frac{2}{D+\eta} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\right] + \left\{\frac{3}{2} \cdot \frac{\eta^2 \cdot D}{(D+\eta)^4} \cdot\right.$$

$$\left.\left[\frac{2}{D+\eta} \left(\frac{\partial\eta}{\partial x}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\right] + \frac{\eta \cdot D^2}{(D+\eta)^5} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2\right\} \cdot z^2$$

The energy-flux, $E_{flux}(\eta)$, per unit length of the wave-front and per unit time through a vertical will, with the bottom, $z = 0$, as reference level, be

$$E_{flux}(\eta) = \int_0^{D+\eta} \left[p(z,t) + \gamma \cdot z + \frac{\gamma}{2 \cdot g} \cdot (u^2(z) + w^2(z)) \right] \cdot u(z) \cdot dz = \gamma \cdot C \cdot \eta \cdot (D + \eta)$$

$$+ \frac{\gamma}{2 \cdot g} \cdot C^3 \cdot \frac{\eta^3}{(D+\eta)^2} - \frac{\gamma}{6g} \cdot C^3 \cdot D^2 \cdot \frac{\eta}{(D+\eta)^2} \cdot \left(\frac{\partial\eta}{\partial x}\right)^2$$

$$+ \frac{\gamma}{3 \cdot g} \cdot C^3 \cdot D^2 \cdot \frac{\eta}{D+\eta} \cdot \frac{\partial^2\eta}{\partial x^2} \qquad (52')$$

The transported energy per unit length of the wave-front, E_{trans}, is then found by numerical integration:

$$E_{trans} = \int_0^T E_{flux}(\eta) \cdot dt \qquad (53')$$

E_{trans} is not dependent on a reference level because:

$$\int_0^T \int_0^{D+\eta} u(z) \cdot dz \cdot dt = \int_0^T Q \cdot dt = 0$$

The volume of the crest, or of the water above mean water level, per unit width, A , is got by using (12'), (13'), (44'), (45') and (47'):

$$A = 2 \cdot \int_0^{\lambda_0} \eta \cdot dx = 2 \cdot \int_0^{\lambda_0} - |\eta_t| \cdot dx$$

$$+ 2 \cdot \int_0^{\theta_0} H \cdot cn^2\theta \cdot \frac{L}{2 \cdot K(k)} \cdot d\theta =$$

$$- 2 \cdot |\eta_t| \cdot \lambda_0 + \frac{H \cdot L}{k^2 \cdot K(k)} \cdot (E(\phi_0 , k) - k_c^2 \cdot \theta_0) \qquad (54')$$

The total horizontal momentum of the crest, per unit width, I_{fr} , is then by (22'):

$$I_{fr} = \int_{-\lambda_o}^{\lambda_o} \frac{\gamma}{g} \cdot Q \cdot dx = 2 \cdot \frac{\gamma}{g} \cdot C \cdot \int_{0}^{\lambda_o} \eta \cdot dx = \frac{\gamma}{g} \cdot C \cdot A \quad (55')$$

(The total horizontal momentum of the trough is the same).

A CONSIDERATION ON HIGHER ORDER TERMS

In the foregoing the phrase: "higher order term" has been used, where for instance a 3rd order term meant:

$\frac{\partial^3\eta}{\partial x^3}$ or $\frac{\partial\eta}{\partial x} \cdot \frac{\partial^2\eta}{\partial x^2}$ or $(\frac{\partial\eta}{\partial x})^3$. From (8') there is got:

$$\frac{K(k)}{L} = \frac{\sqrt{3}}{4} \cdot \frac{1}{D} \cdot \sqrt{\frac{H}{D}} \cdot \frac{1}{k}$$

k is close to 1 for any cnoidal wave here considered. From (17'), (18') and (19') there can then be extracted the sizes determining the maximum values of importance in comparison of different orders:

$$\frac{\partial\eta}{\partial x} \sim \frac{H}{D} \cdot \sqrt{\frac{H}{D}}$$

$$\frac{\partial^2\eta}{\partial x^2} \sim \frac{1}{D} \cdot (\frac{H}{D})^2$$

$$\frac{\partial^3\eta}{\partial x^3} \sim \frac{1}{D^2} \cdot (\frac{H}{D})^2 \cdot \sqrt{\frac{H}{D}}$$

It is seen that the increase in derivation has to be 2 for the increase in power of $\frac{H}{D}$ to be 1. For $\frac{H}{D}$ of more than an insignificant size this yields a rather slow decrease of importance of higher order terms. Of this reason it is advisable to use basic hydrodynamic equations when calculating a new important size in the wave (like in this paper u(z) and p(z)) instead of just returning to some previous equation from the development of the theory. It is also seen that for instance $(\frac{\partial\eta}{\partial x})^3 \sim (\frac{H}{D})^4 \cdot \sqrt{\frac{H}{D}}$, a somewhat higher power than for $\frac{\partial^3\eta}{\partial x^3}$. But as $\frac{H}{D}$ grows towards 1 an exact calculation of the maximum values will show $(\frac{\partial\eta}{\partial x})^3$ to be of the same size as $\frac{\partial^3\eta}{\partial x^3}$. And as it is the purpose to use this theory for waves of any non-breaking height $(\frac{\partial\eta}{\partial x})^3$ must be taken along in a 3rd order expression.

TRANSITION FROM CNOIDAL TO SINUSOIDAL THEORY

It will now be shown, that for $k \to 0$ the cnoidal formulæ will approach the usual sinusoidal formulæ.

The complete elliptic integrals, K(k) and E(k) , can be expanded in powers of k as:

$$K(k) = \frac{\pi}{2} \cdot \left[1 + (\frac{1}{2})^2 \cdot k^2 + (\frac{3}{2 \cdot 4})^2 \cdot k^4 + \sum_{m=3}^{\infty} (\frac{(2 \cdot m)!}{(m!)^2 \cdot 4^m})^2 \cdot k^{2 \cdot m}\right] \quad (56')$$

$$E(k) = \frac{\pi}{2} \cdot \left[1 - (\frac{1}{2})^2 \cdot k^2 - (\frac{3}{2 \cdot 4})^2 \cdot \frac{k^4}{3} - \sum_{m=3}^{\infty} (\frac{(2 \cdot m)!}{(m!)^2 \cdot 4^m})^2 \cdot \frac{k^{2 \cdot m}}{2 \cdot m - 1}\right] \quad (57')$$

Or $K(k)$ and $E(k)$ can be expanded in powers of k_c as:

$$K(k) = \ln \frac{4}{k_c} + \left(\frac{1}{2}\right)^2 \cdot \left(\ln \frac{4}{k_c} - 1\right) \cdot k_c^2 + \left(\frac{3}{2 \cdot 4}\right)^2$$

$$\cdot \left(\ln \frac{4}{k_c} - 1 - \frac{1}{2 \cdot 3}\right) \cdot k_c^4 + \sum_{m=3}^{\infty} \left(\frac{(2 \cdot m)!}{(m!)^2 \cdot 4^m}\right)^2$$

$$\cdot \left(\ln \frac{4}{k_c} - \sum_{m=1}^{m} \frac{1}{n \cdot (2 \cdot n - 1)}\right) \cdot k_c^{2 \cdot m} \qquad (58')$$

$$E(k) = 1 + \frac{1}{2} \cdot \left(\ln \frac{4}{k_c} - \frac{1}{2}\right) \cdot k_c^2 + \left(\frac{1}{2}\right)^2 \cdot \frac{3}{4}$$

$$\cdot \left(\ln \frac{4}{k_c} - 1 - \frac{1}{3 \cdot 4}\right) \cdot k_c^4 + \sum_{m=3}^{\infty} \left(\frac{(2 \cdot (m - 1))!}{((m - 1)!)^2 \cdot 4^{m-1}}\right)^2$$

$$\cdot \frac{2 \cdot m - 1}{2 \cdot m} \cdot \left(\ln \frac{4}{k_c} - \sum_{n=1}^{m-1} \frac{1}{n \cdot (2 \cdot n - 1)}\right.$$

$$\left. - \frac{1}{2 \cdot m \cdot (2 \cdot m - 1)}\right) \cdot k_c^{2 \cdot m} \qquad (59')$$

From (8') it is got:

$$\frac{1}{k^2} = \frac{16}{3} \cdot \frac{D^3}{H \cdot L^2} \cdot K^2(k) \qquad (60')$$

This is used in (7') together with $k = 0$ and the first term in (56') and (57'):

$$\frac{c^2}{g \cdot D} = 1 + \frac{16}{3} \cdot \frac{D^2 \cdot K^2(k)}{L^2} \cdot \left(2 - 3 \cdot \frac{E(k)}{K(k)} - k^2\right)$$

$$= 1 - \frac{4}{3} \cdot \frac{\pi^2 \cdot D^2}{L^2}$$

This is the same expression as the first two terms in the power expansion of the sinusoidal celerity:

$$C^2 = \frac{g \cdot L}{2 \cdot \pi} \cdot \tanh \frac{2 \cdot \pi \cdot D}{L} = g \cdot D \cdot \left[1 - \frac{4}{3} \cdot \frac{\pi^2 \cdot D^2}{L^2} + \cdots\right] \qquad (61')$$

From (9') it is seen, that if T is finite, then $k \to 0$ demands $H \to 0$. The same is demanded of the wave-height in the development of the sinusoidal theory.

Using the first two terms in (56') and (57') and letting $k \to 0$, η_c in (10') changes to:

$$\eta_c = \frac{H}{k^2} \cdot \left(1 - \frac{\frac{\pi}{2} \cdot (1 - \frac{1}{4} \cdot k^2)}{\frac{\pi}{2} \cdot (1 + \frac{1}{4} \cdot k^2)}\right) = \frac{H}{k^2} \cdot \frac{1 + \frac{1}{4} \cdot k^2 - 1 + \frac{1}{4} \cdot k^2}{1 + \frac{1}{4} \cdot k^2}$$

$$\to \frac{H}{2} \qquad (62')$$

The surface profile of the sinusoidal waves can be written as:

$$\eta = \frac{H}{2} \cdot \cos\left[2 \cdot \pi \cdot \left(\frac{x}{L} - \frac{t}{T}\right)\right] = -\frac{H}{2} + H \cdot \cos^2\left[\pi \cdot \left(\frac{x}{L} - \frac{t}{T}\right)\right] \qquad (63')$$

This is the same as got by (12') for $k \to 0$.

Regarding η and $\left(\frac{\partial \eta}{\partial x}\right)^2$ as comparatively small the horizontal particle velocity from (29') can be written as:

$$u(z) = C \cdot \left(\frac{\eta}{D} - \frac{3 \cdot z^2 - D^2}{6 \cdot D} \cdot \frac{\partial^2 \eta}{\partial x^2}\right) \qquad (64')$$

With $R = \frac{2 \cdot \pi}{L}$ (63') gives:

$$- \frac{\partial^2 \eta}{\partial x^2} = R^2 \cdot \eta \qquad (65')$$

Thereby (64') can be written as:

$$u(z) = C \cdot \frac{\eta}{D} (1 + \frac{1}{2} \cdot R^2 \cdot z^2 - \frac{1}{6} \cdot R^2 \cdot D^2)$$

This is the same as yielded by the first two terms in the power expansion of the sinusoidal expression:

$$u(z) = \frac{2 \cdot \pi}{T} \cdot \frac{\cosh(R \cdot z)}{\sinh(R \cdot D)} \cdot \eta \simeq C \cdot R \cdot \eta$$

$$\cdot \frac{1 + \frac{1}{2} \cdot R^2 \cdot z^2}{R \cdot D + \frac{1}{6} \cdot R^3 \cdot D^3} \simeq C \cdot \frac{\eta}{D} \cdot (1 + \frac{1}{2} \cdot R^2 \cdot z^2 - \frac{1}{6} \cdot R^2 \cdot D^2)$$

Regarding η and $(\frac{\partial \eta}{\partial x})^3$ and $\frac{\partial \eta}{\partial x} \cdot \frac{\partial^2 \eta}{\partial x^2}$ as comparatively small the vertical particle velocity from (30') can be written as:

$$w(z) = C \cdot \left[- z \cdot (\frac{1}{D} \cdot \frac{\partial \eta}{\partial x} + \frac{D}{6} \cdot \frac{\partial^3 \eta}{\partial x^3}) + \frac{z^3}{6 \cdot D} \cdot \frac{\partial^3 \eta}{\partial x^3}\right]$$

Like in (65') there is got:

$$- \frac{\partial^3 \eta}{\partial x^3} = R^2 \cdot \frac{\partial \eta}{\partial x}$$

whereby $w(z)$ can be written as:

$$w(z) = C \cdot \frac{\partial \eta}{\partial x} \cdot \frac{z}{D} \cdot (- 1 + \frac{1}{6} \cdot R^2 \cdot D^2 - \frac{1}{6} \cdot R^2 \cdot z^2)$$

This is the same as yielded by the first two terms of the power expansion of the sinusoidal expression:

$$w(z) = -\frac{2\pi}{T} \cdot \frac{\sinh(R \cdot z)}{\sinh(R \cdot D)} \cdot \frac{1}{R} \cdot \frac{\partial\eta}{\partial x} \simeq -C \cdot \frac{\partial\eta}{\partial x} \cdot$$

$$\frac{R \cdot z + \frac{1}{6} \cdot R^3 \cdot z^3}{R \cdot D + \frac{1}{6} \cdot R^3 \cdot D^3} \simeq C \cdot \frac{\partial\eta}{\partial x} \cdot \frac{z}{D} \cdot (-1 + \frac{1}{6} \cdot R^2 \cdot D^2$$

$$- \frac{1}{6} \cdot R^2 \cdot z^2)$$

Regarding η and $(\frac{\partial\eta}{\partial x})^2$ as comparatively small and $C^2 = g \cdot D$ the wave-pressure from (40') can be written as, using (65'):

$$p^+(z) = \gamma \cdot \eta + \frac{\gamma}{2} \cdot \frac{\partial^2\eta}{\partial x^2} \cdot (D^2 - z^2)$$

$$= \gamma \cdot \eta \cdot (1 - \frac{1}{2} \cdot R^2 \cdot D^2 + \frac{1}{2} \cdot R^2 \cdot z^2)$$

This is the same as yielded by the first two terms in the power expansion of the sinusoidal expression:

$$p^+(z) = \gamma \cdot \eta \cdot \frac{\cosh(R \cdot z)}{\cosh(R \cdot D)} \simeq \gamma \cdot \eta \cdot \frac{1 + \frac{1}{2} \cdot R^2 \cdot z^2}{1 + \frac{1}{2} \cdot R^2 \cdot D^2}$$

$$\simeq \gamma \cdot \eta \cdot (1 - \frac{1}{2} \cdot R \cdot D^2 + \frac{1}{2} \cdot R^2 \cdot z^2)$$

For $k \to 0$ the different terms in the potential energy (49') will do as follows. Using (62') it is got:

$$\frac{\eta_t^2}{2} \to \frac{H^2}{8}$$

With $k_c^2 = 1 - k^2$ and using the first two terms in (56') and (57') it is got:

$$- H \cdot |\eta_t| \cdot \frac{E(k) - k_c^2 \cdot K(k)}{k^2 \cdot K(k)} \to - \frac{1}{4} \cdot H^2$$

With $k_c^2 = 1 - k^2$ and using the first 3 terms in (56') and (57') it is got:

$$\frac{H^2}{6} \cdot \frac{(2 - 3 \cdot k^2) \cdot k_c^2 \cdot K(k) + 2 \cdot (2 \cdot k^2 - 1) \cdot E(k)}{k^4 \cdot K(k)}$$

$$\to \frac{3}{16} \cdot H^2$$

Hereby it is got as wanted:

$$E_{pot} \to \frac{\gamma}{16} \cdot H^2$$

Regarding η as comparatively small the kinetic energy can be expressed as, using $\Delta E_{kin}(\eta)$ from (50'):

$$\Delta E_{kin}(\eta) = \frac{\gamma \cdot c^2}{2 \cdot g} \cdot \left[\frac{\eta^2}{D} + \frac{1}{3} \cdot D \cdot (\frac{\partial \eta}{\partial x})^2\right]$$

The same is got from the sinusoidal expression:

$$\Delta E_{kin}(\eta) = \int_0^D \frac{\gamma}{2 \cdot g} \cdot (u^2(z) + w^2(z)) \cdot dz =$$

$$\frac{\gamma}{2 \cdot g} \cdot \int_0^D c^2 \cdot \frac{1}{\sinh^2(R \cdot D)} \cdot \Big[\cosh^2(R \cdot z) \cdot R^2 \cdot \eta^2$$

$$+ \sinh^2(R \cdot z) \cdot (\frac{\partial \eta}{\partial x})^2\Big] \cdot dz = \frac{\gamma}{4 \cdot g} \cdot c^2 \cdot \Big[R \cdot \eta^2$$

$$\cdot (\frac{\cosh(R \cdot D)}{\sinh(R \cdot D)} + \frac{R \cdot D}{\sinh^2(R \cdot D)}) + \frac{1}{R} \cdot (\frac{\partial \eta}{\partial x})^2 \cdot (\frac{\cosh(R \cdot D)}{\sinh(R \cdot D)}$$

$$- \frac{R \cdot D}{\sinh^2(R \cdot D)})\Big] \simeq \frac{\gamma \cdot c^2}{2 \cdot g} \cdot \Big[\frac{\eta^2}{D} + \frac{1}{3} \cdot D \cdot (\frac{\partial \eta}{\partial x})^2\Big]$$

where the hyperbolic terms were expanded as follows:

$$\frac{\cosh(R \cdot D)}{\sinh(R \cdot D)} \simeq \frac{1}{R \cdot D} \cdot (1 + \frac{1}{3} \cdot R^2 \cdot D^2)$$

$$\frac{1}{\sinh^2(R \cdot D)} \simeq \frac{1}{R^2 \cdot D^2} \cdot (1 - \frac{1}{3} \cdot R^2 \cdot D^2)$$

Regarding η and $(\frac{\partial \eta}{\partial x})^2$ as comparatively small and $c^2 = g \cdot D$ the energy-flux (52') can be written as, using (65'):

$$E_{flux}(\eta) = \gamma \cdot c \cdot \eta \cdot D + \gamma \cdot c \cdot \eta^2$$

$$+ \frac{\gamma}{2 \cdot g} \cdot c^3 \cdot \frac{\eta^3}{D^2} - \frac{1}{3} \cdot \gamma \cdot c \cdot D^2 \cdot R^2 \cdot \eta^2$$

The same is got from the sinusoidal expression using

$$\left(\frac{\partial \eta}{\partial x}\right)^2 \simeq 0$$

$$E_{flux}(\eta) = \int_0^D \left[p(z) + \gamma(z) + \frac{\gamma}{2 \cdot g} \cdot (u^2(z) + w^2(z))\right]$$

$$\cdot\ u(z) \cdot dz = \int_0^D \left[\gamma \cdot D + \gamma \cdot \eta \cdot \frac{\cosh(R \cdot z)}{\cosh(R \cdot D)}\right.$$

$$+ \frac{\gamma}{2 \cdot g} \cdot C^2 \cdot \frac{1}{\sinh^2(R \cdot D)} \cdot (R^2 \cdot \eta^2 \cdot \cosh^2(R \cdot z)$$

$$\left. + \left(\frac{\partial \eta}{\partial x}\right)^2 \cdot \sinh^2(R \cdot z))\right] \cdot C \cdot R \cdot \eta \cdot \frac{\cosh(R \cdot z)}{\sinh(R \cdot D)} \cdot dz$$

$$= \gamma \cdot C \cdot \eta \cdot D + \gamma \cdot C \cdot \eta^2 - \frac{1}{3} \cdot \gamma \cdot C \cdot \eta^2 \cdot R^2 \cdot D^2$$

$$+ \frac{\gamma}{2 \cdot g} \cdot C^3 \cdot \frac{\eta^3}{D^2}$$

THE SOLITARY WAVE

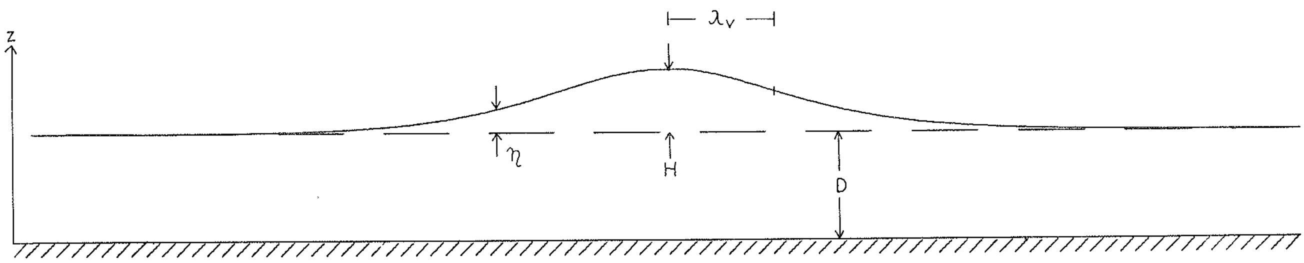


Fig.2. Definition sketch. Solitary wave. Undistorted.

H/D = 0.6

The solitary wave can be found from eq. 35 by demanding $\eta \geq 0$. But the same solution will be got from the cnoidal wave by letting $k \rightarrow 1$. As wanted (8') then, and for $\frac{H}{D} > 0$ only then, gives:

$$\frac{L}{D} = \frac{4}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot k \cdot K(k) \rightarrow \infty \tag{66'}$$

(7') and (10') give :

$$\frac{C^2}{g \cdot D} = 1 + \frac{H}{D} \tag{67'}$$

$$\eta_c = H \tag{68'}$$

Using $\frac{2 \cdot K(k)}{L}$ from (8') in (13') we get

$$\theta = \frac{\sqrt{3}}{2 \cdot D} \cdot \sqrt{\frac{H}{D}} \cdot (x - C \cdot t) \tag{69'}$$

and the surface profile of (12'):

$$\eta = \frac{H}{\cosh^2 \theta} = H \cdot \operatorname{sech}^2 \theta \tag{70'}$$

The derivatives of (70') are the same as (17'), (18') and (19') would give by $k = 1$:

$$\frac{\partial \eta}{\partial x} = -\sqrt{3} \cdot \frac{H}{D} \cdot \sqrt{\frac{H}{D}} \cdot \frac{\sinh \theta}{\cosh^3 \theta} \tag{71'}$$

$$\frac{\partial^2 \eta}{\partial x^2} = -\frac{3}{2} \cdot \frac{H^2}{D^3} \cdot \left(\frac{3}{\cosh^4 \theta} - \frac{2}{\cosh^2 \theta}\right) \tag{72'}$$

$$\frac{\partial^3 \eta}{\partial x^3} = 3 \cdot \sqrt{3} \cdot \frac{H^2}{D^4} \cdot \sqrt{\frac{H}{D}} \cdot \left(\frac{3 \cdot \sinh \theta}{\cosh^5 \theta} - \frac{\sinh \theta}{\cosh^3 \theta}\right) \tag{73'}$$

Using (70'), (71'), (72') and (73') the particle velocities and accel erations and the pressure are again found from (29') to (40'). The point of inflection is found from (72') with $\frac{\partial^2 \eta}{\partial x^2} = 0$:

$$\cosh^2 \theta_v = \frac{3}{2} \qquad (74')$$

and with $x = 0$ at the crest (69') gives:

$x = \lambda_v$ at θ_v

$$\lambda_v = \frac{2 \cdot D}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot \ln \frac{\sqrt{3} + 1}{\sqrt{2}} \qquad (75')$$

where:

$$\ln \frac{\sqrt{3} + 1}{\sqrt{2}} = 0.6585 \simeq \frac{2}{3} \qquad (76')$$

(74') in (70') gives:

$$\eta_v = \frac{2}{3} \cdot H \qquad (77')$$

The horizontal distance from the crest to the point of inflection, λ_v, is the dynamically important length in the solitary and cnoidal waves and not the wave-length, or the distance from the crest to the mean water level. Some co-ordinates $(x - C \cdot t, \eta)$ of the surface are: $(2 \cdot \lambda_v, \frac{1}{4} \cdot H)$, $(3 \cdot \lambda_v, \frac{2}{27} \cdot H)$ and $(4 \cdot \lambda_v, \frac{1}{49} \cdot H)$

The total potential energy, E_{pot}, per unit width will be, using (69') and (70'):

$$E_{pot} = \int_{-\infty}^{\infty} \frac{\gamma}{2} \cdot \eta^2 \cdot dx = \gamma \cdot \frac{2 \cdot D}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot H^2 \cdot \int_0^{\infty} \frac{d\theta}{\cosh^4 \theta}$$

$$= \frac{2}{\sqrt{3}} \cdot \gamma \cdot H \cdot D \cdot \sqrt{H \cdot D} \cdot \left[\frac{2}{3} \cdot \tanh\theta \cdot \left(\frac{1}{2 \cdot \cosh^2\theta} + 1\right)\right]_0^{\infty}$$

$$= \frac{4}{3 \cdot \sqrt{3}} \cdot \gamma \cdot H \cdot D \cdot \sqrt{H \cdot D} \qquad (78')$$

This is the same value as would be obtained from (49').
Using (75') and (76') E_{pot} can be written as:

$$E_{pot} = \frac{\gamma}{2} \cdot H^2 \cdot \frac{2 \cdot \lambda_v}{\ln \frac{\sqrt{3} + 1}{\sqrt{2}}} \cdot \frac{2}{3} \simeq \frac{\gamma}{2} \cdot H^2 \cdot 2 \cdot \lambda_v$$

which yields, that the total potential energy of the solitary wave is the same as that of a rectangular mass of water of height H, placed between the two points of inflection.

The kinetic energy, ΔE_{kin}, will be the same as in (50'). To calculate (51') are needed some integrals. a is a constant, Θ and θ are variables:

$$\int \frac{1}{1 + a - \Theta^2} \cdot d\Theta = \frac{1}{2 \cdot \sqrt{1 + a}} \cdot \ln \frac{\sqrt{1 + a} + \Theta}{\sqrt{1 + a} - \Theta}$$

$$\int \frac{\Theta^2}{1 + a - \Theta^2} \cdot d\Theta = -\Theta + (1 + a) \cdot \int \frac{1}{1 + a - \Theta^2} \cdot d\Theta$$

$$\int \frac{\Theta^4}{1 + a - \Theta^2} \cdot d\Theta = -\frac{\Theta^3}{3} + (1 + a) \cdot \int \frac{\Theta^2}{1 + a - \Theta^2} \cdot d\Theta$$

$$\int \frac{\operatorname{sech}^4\theta}{a + \operatorname{sech}^2\theta} \cdot d\theta = \int \frac{1 - \tanh^2\theta}{a + 1 - \tanh^2\theta} \cdot d \tanh\theta$$

$$\int \frac{\operatorname{sech}^4\theta \cdot \tanh^2\theta}{a + \operatorname{sech}^2\theta} \cdot d\theta = \int \frac{(1 - \tanh^2\theta) \cdot \tanh^2\theta}{a + 1 - \tanh^2\theta} \cdot d \tanh\theta$$

The total kinetic energy, E_{kin}, per unit width will then be, using (50'), (69'), (70') and (71'):

$$E_{kin} = \int_{-\infty}^{\infty} \Delta E_{kin}(\eta) \cdot dx = \frac{\gamma \cdot C^2}{g} \cdot \int_0^{\infty} \frac{\eta^2}{D + \eta} \cdot dx$$

$$+ \frac{\gamma \cdot D^2 \cdot C^2}{3 \cdot g} \cdot \int_0^{\infty} \frac{1}{D + \eta} \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 \cdot dx$$

$$= \frac{\gamma \cdot C^2 \cdot H}{g} \cdot \frac{2 \cdot D}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot \int_0^{\infty} \frac{\operatorname{sech}^4 \theta}{\frac{D}{H} + \operatorname{sech}^2 \theta} \cdot d\theta$$

$$+ \frac{\gamma \cdot C^2 \cdot H^2}{g} \cdot \frac{2}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot \int_0^{\infty} \frac{\operatorname{sech}^4 \theta \cdot \tanh^2 \theta}{\frac{D}{H} + \operatorname{sech}^2 \theta} \cdot d\theta$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\gamma}{g} \cdot C^2 \cdot \sqrt{H \cdot D} \cdot \left\{ D \cdot \left[\tanh \theta - \frac{\frac{D}{H}}{2 \cdot \sqrt{1 + \frac{D}{H}}} \right.\right.$$

$$\left. \cdot \ln \frac{\sqrt{1 + \frac{D}{H}} + \tanh \theta}{\sqrt{1 + \frac{D}{H}} - \tanh \theta} \right]_0^{\infty} + H \cdot \left[\frac{1}{3} \cdot \tanh^3 \theta + \frac{D}{H} \cdot \tanh \theta \right.$$

$$\left.\left. - \frac{D}{2 \cdot H} \cdot \sqrt{1 + \frac{D}{H}} \cdot \ln \frac{\sqrt{1 + \frac{D}{H}} + \tanh \theta}{\sqrt{1 + \frac{D}{H}} - \tanh \theta} \right]_0^{\infty} \right\}$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\gamma}{g} \cdot C^2 \cdot \sqrt{H \cdot D} \cdot \left\{ D \cdot \left[1 - \frac{\frac{D}{H}}{2 \cdot \sqrt{1 + \frac{D}{H}}} \right.\right.$$

$$\left. \cdot \ln \frac{\sqrt{1 + \frac{D}{H}} + 1}{\sqrt{1 + \frac{D}{H}} - 1} \right] + H \cdot \left[\frac{1}{3} + \frac{D}{H} - \frac{D}{2 \cdot H} \right.$$

$$\left.\left. \cdot \sqrt{1 + \frac{D}{H}} \cdot \ln \frac{\sqrt{1 + \frac{D}{H}} + 1}{\sqrt{1 + \frac{D}{H}} - 1} \right] \right\} \qquad (79')$$

As could be expected $E_{kin} > E_{pot}$

The volume per unit width of the wave, A , is, using (69') and (70') :

$$A = \int_{-\infty}^{\infty} \eta \cdot dx = 2 \cdot \frac{2 \cdot D}{\sqrt{3}} \cdot \sqrt{\frac{D}{H}} \cdot H \cdot \int_{0}^{\infty} \frac{d\theta}{\cosh^2 \theta}$$

$$= \frac{4 \cdot D}{\sqrt{3}} \cdot \sqrt{H \cdot D} \qquad (80')$$

The total horizontal momentum is then, using (22'):

$$I_{fr} = \int_{-\infty}^{\infty} \frac{\gamma}{g} \cdot Q \cdot dx = \frac{\gamma}{g} \cdot C \cdot \int_{-\infty}^{\infty} \eta \cdot dx$$

$$= \frac{4}{\sqrt{3}} \cdot \frac{\gamma}{g} \cdot C \cdot D \cdot \sqrt{H \cdot D}$$

COMPARISON OF THEORY WITH MODEL TESTS

The profile and celerity of the solitary wave as given by (70') and (67') has been verified by several investigators. As the solitary wave and the sinusoidal wave are the extremes of the cnoidal wave there are good reasons to accept the cnoidal expressions from (7') to (13').

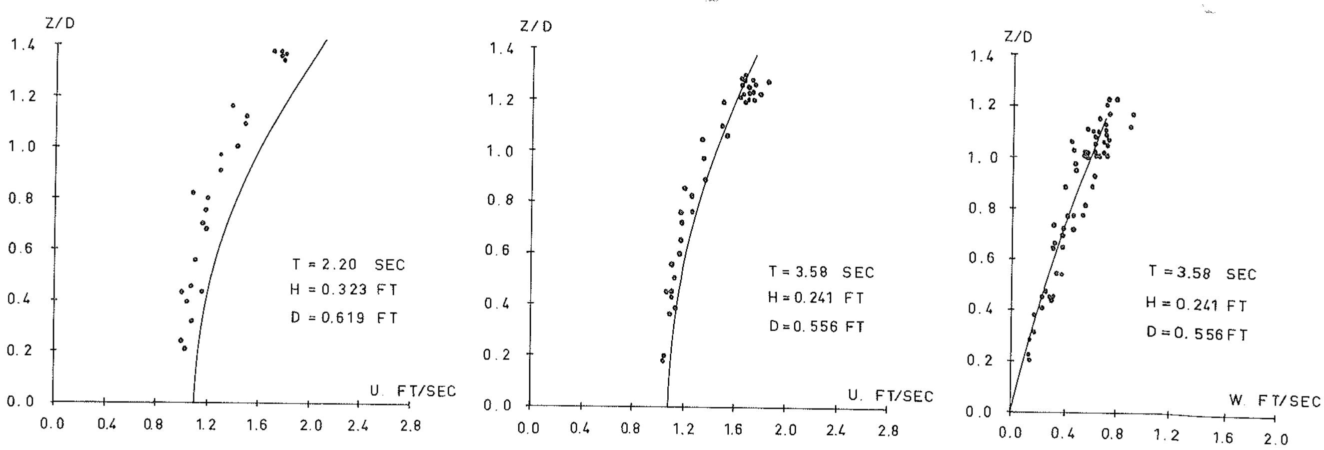


Fig.3. Comparison of theory with model tests for the maximum values of particle velocities, U horizontal and W vertical. Experiments from Mehauté et al. , reference [13]

The new expressions for particle velocities (29') and (30') have been compared to tests in [13], and here is shown the best and the poorest result. For comparison with other theories see [13].

As a further control of the velocity expression the Bernoulli equation can be used on the free surface. For sinusoidal waves this gives the wanted proof. For the other extreme, the solitary wave, we get

$$2 \cdot C \cdot U_{top} - U_{top}^2 = 2 \cdot g \cdot H$$

where U_{top} is the particle velocity at the surface of the crest. This gives a maximum disagreement of 7% (for H/D = 0.8).

APPLICATION OF THE GRAPHS AND THE CHOICE BETWEEN THEORIES

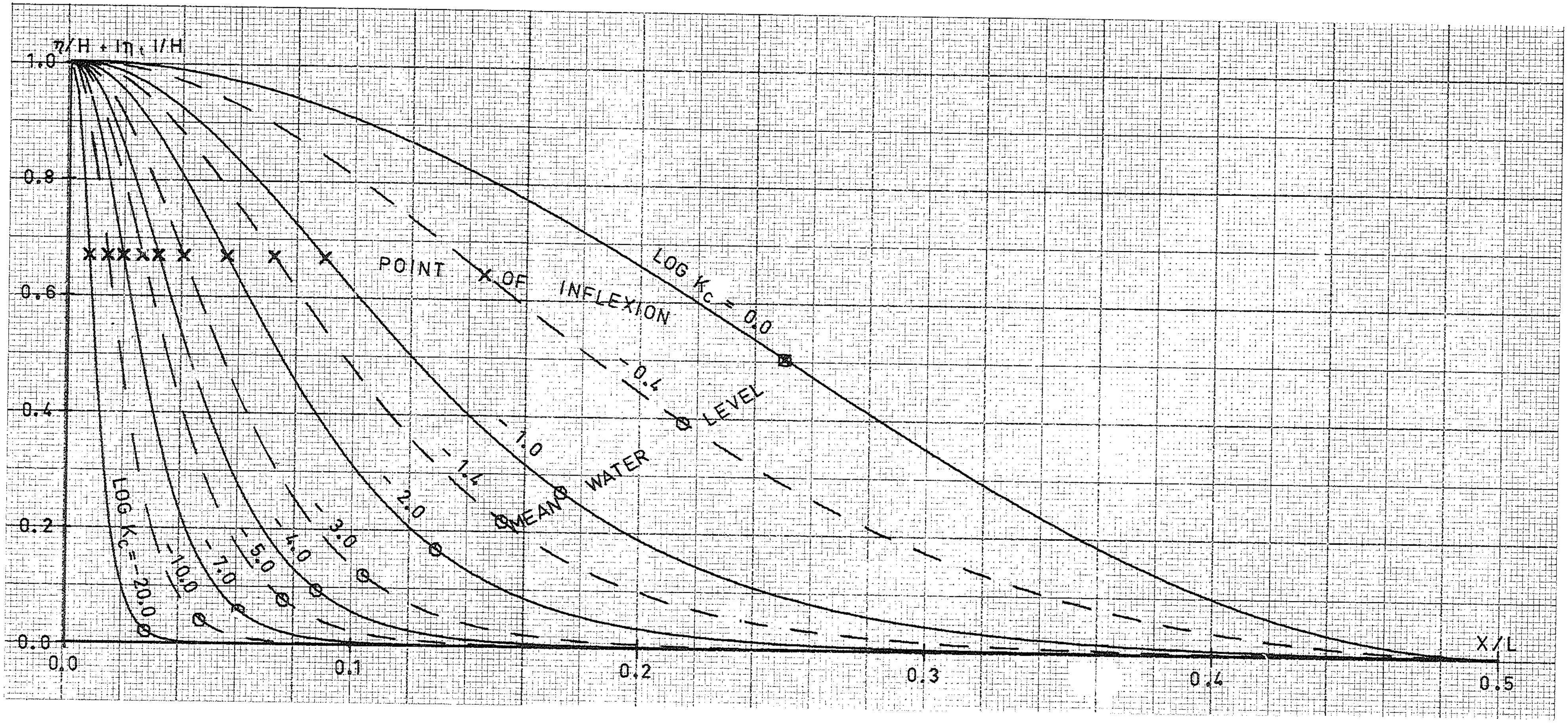
Of technical reasons there are used only capital letters in the latin alphabet on the computer drawn graphs, so that for instance the parameter log k_c is written LOG K_C. The different symbols are explained in the NOTATION. Note that for the acceleration term 1 metre/second^2 there is written only: 1. For the particle velocities and the pressure the formulæ must be used with the necessary sizes found from the graphs. When there is a deviation between the background millimeter net and the computer drawn marks, the computer marks should of course be used.

When getting outside of the cnoidal graphs because - log k_c gets too great, the graphs for the solitary wave should be used. Or if greater accuracy is wanted, the formulæ can be used with: k = 1, E(k) = 1, $K(k) = \ln \frac{4}{k_c}$ (see (58') and (59')), and $\mathrm{cn}\,\Theta = \cosh \Theta$.

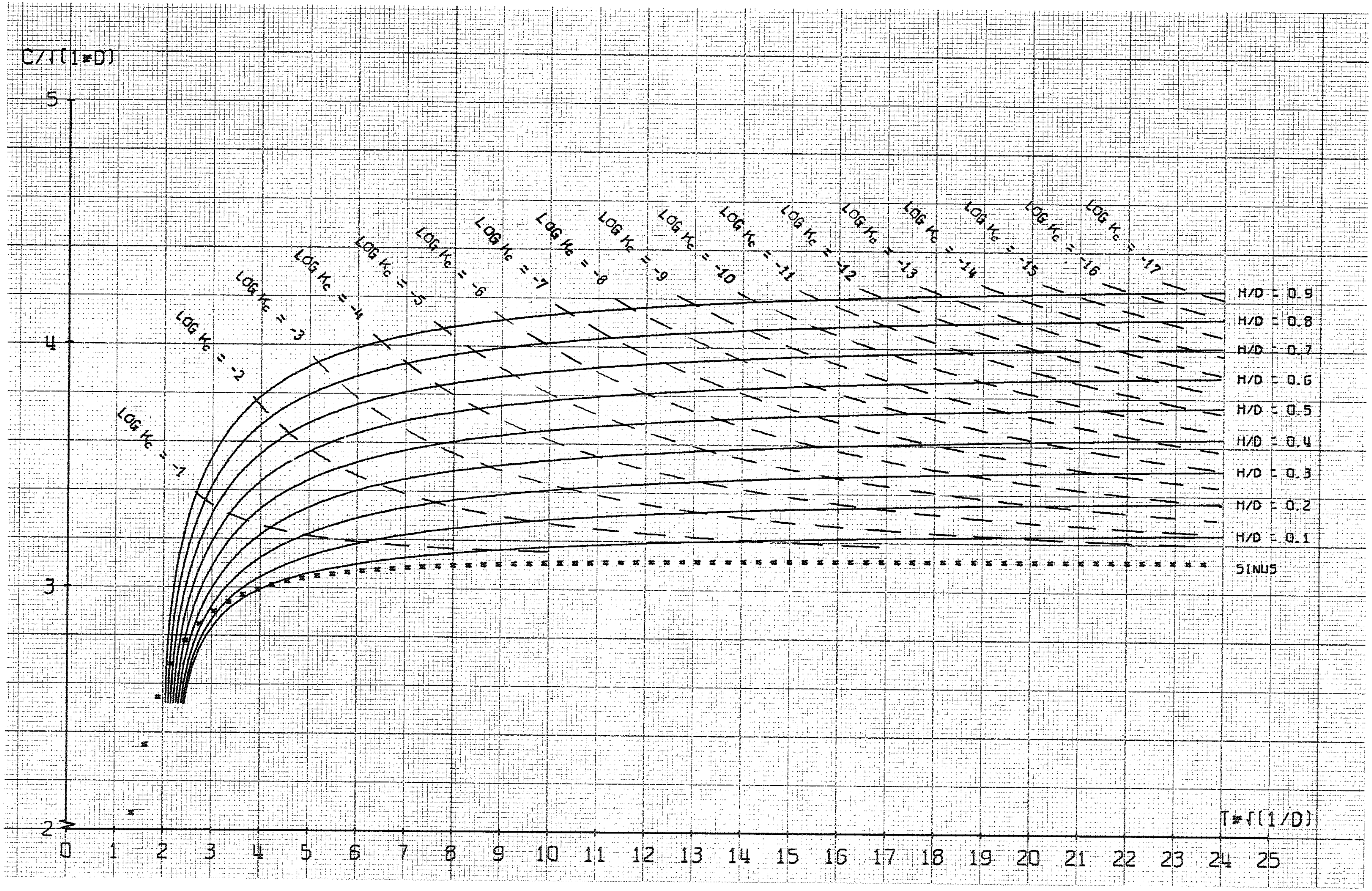
At the other extreme the sinusoidal waves should be used. The celerity, C_{sine} according to the Stokes' sinusoidal theory of first and second order is:

$$C_{sine} = \sqrt{\frac{g \cdot L_{sine}}{2 \cdot \pi} \tanh \left(\frac{2 \cdot \pi \cdot D}{L_{sine}}\right)}$$

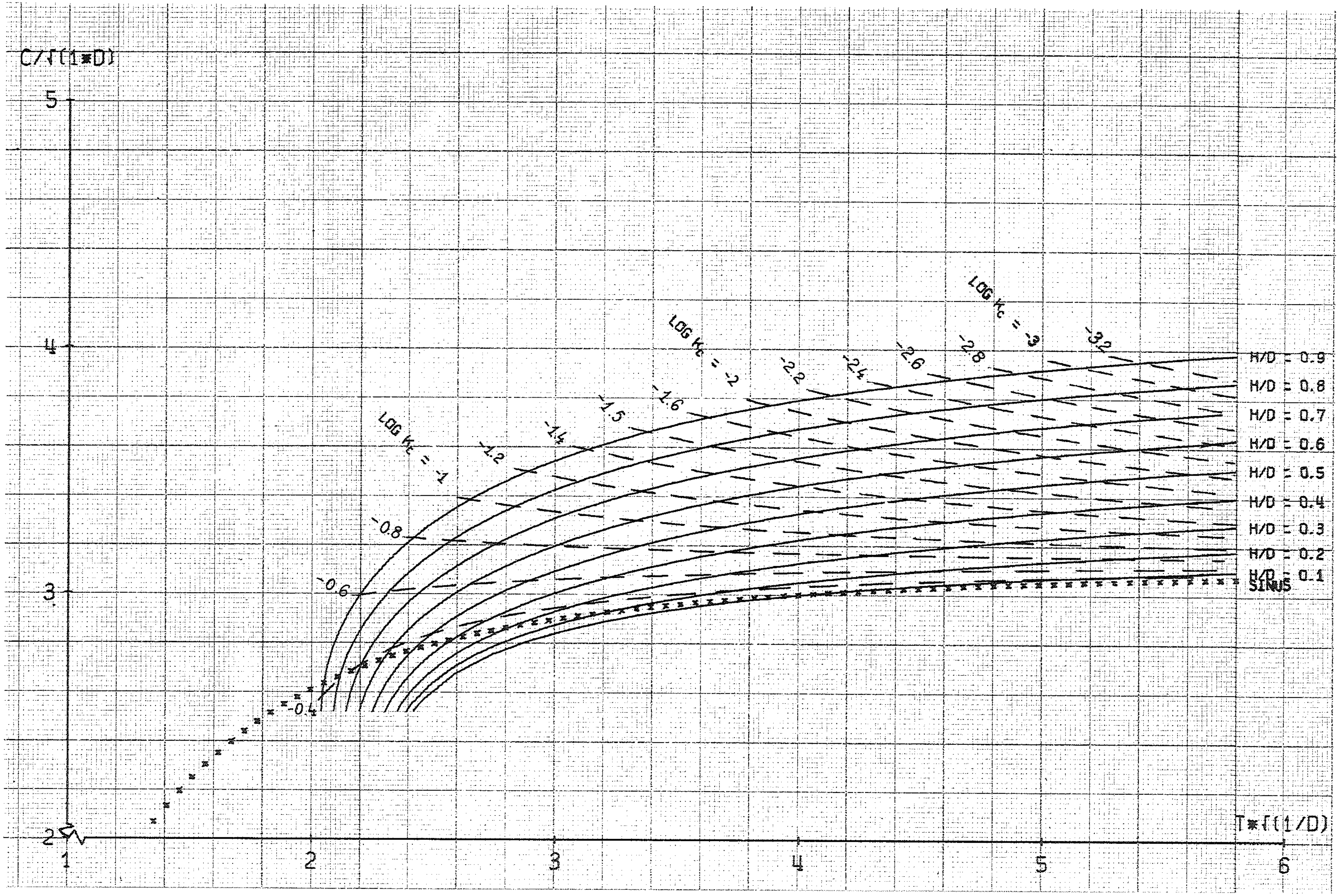
C_{sine} is also plotted out, and it is proposed to use cnoidal expressions when $C_{cnoi} > C_{sine}$. In this way there is got a continuous transition between the two theories for C and L for given continuous T and D. For η_c a discontinuous transition is got, but the discontinuity is of no more than 23 % (for H/D = o.9). The cnoidal value of η_c in the transition lies between the values of the first and second order Stokes' theory and it is reasonable only for small values of H/D ($\sim$ o.1) to use the second order theory for better transition.



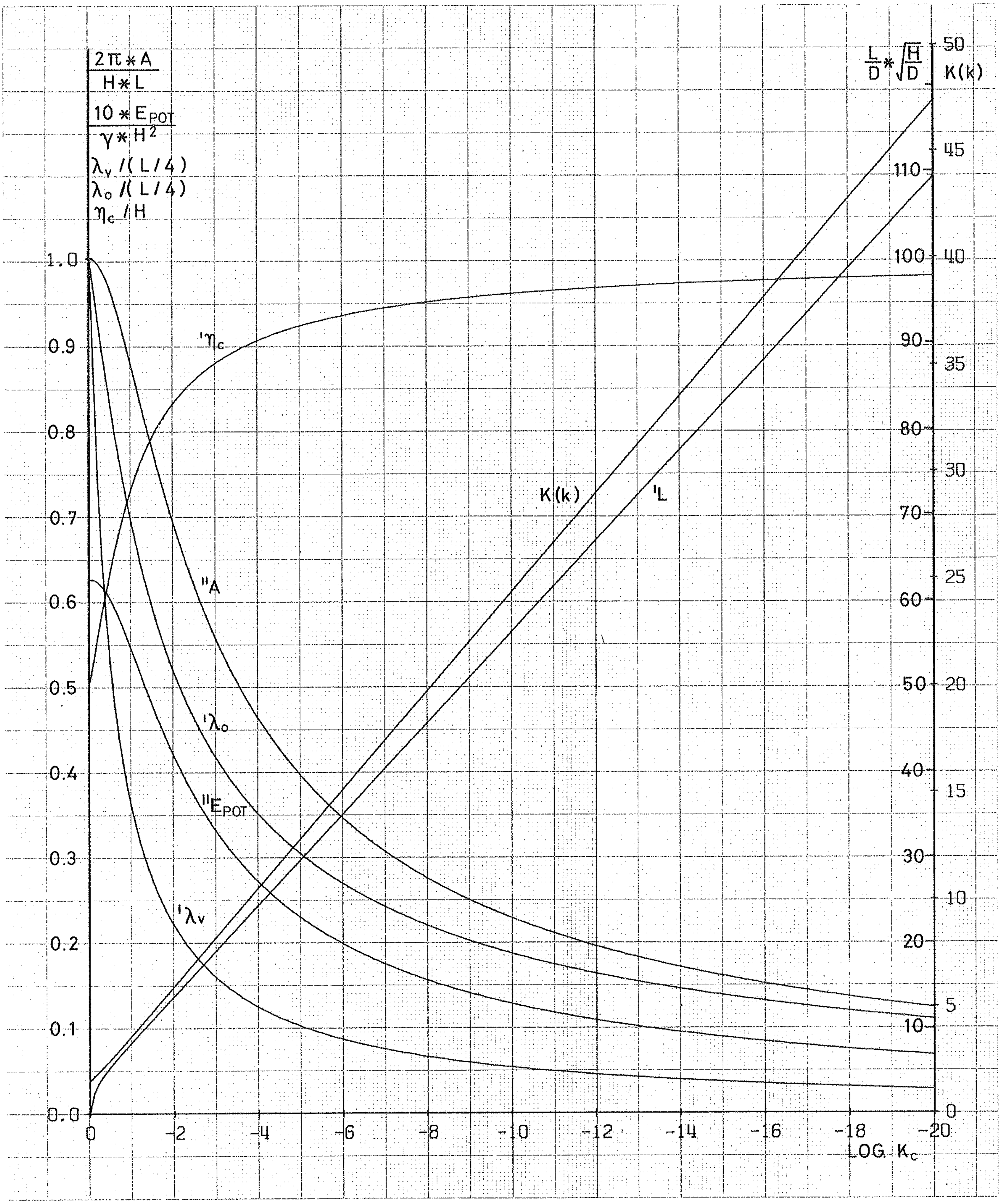
GRAPH 1. Distorted cnoidal wave profile for different values of the parameter log k_c. In formulæ for particle velocies and accelerations and for pressure use: $\eta/H - |\eta|/H = cn^2\theta$.



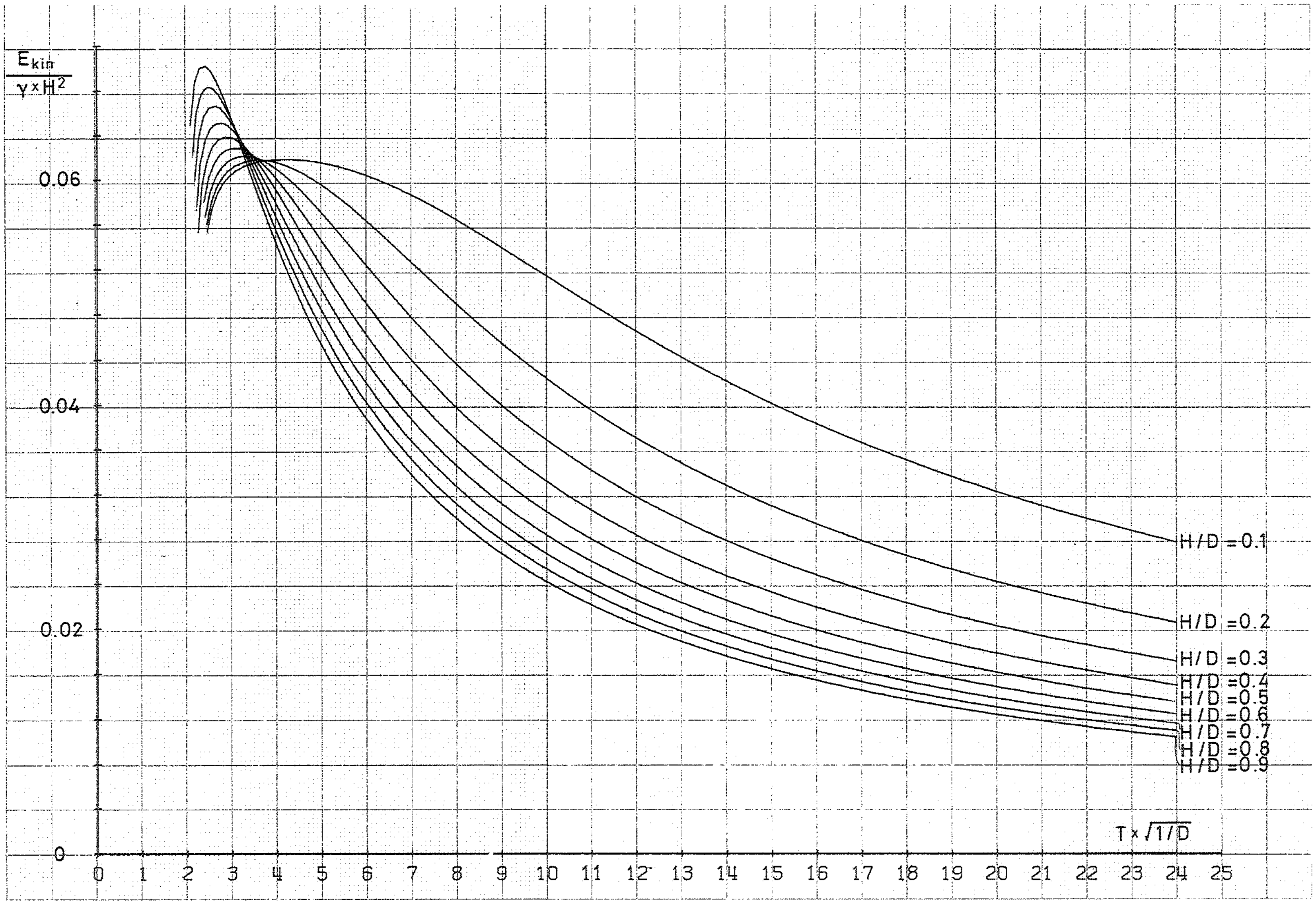
GRAPH 2. Cnoidal wave. The celerity and log k_c as functions of the wave-period and the mean water depth. (1 is 1 m/sec^2)



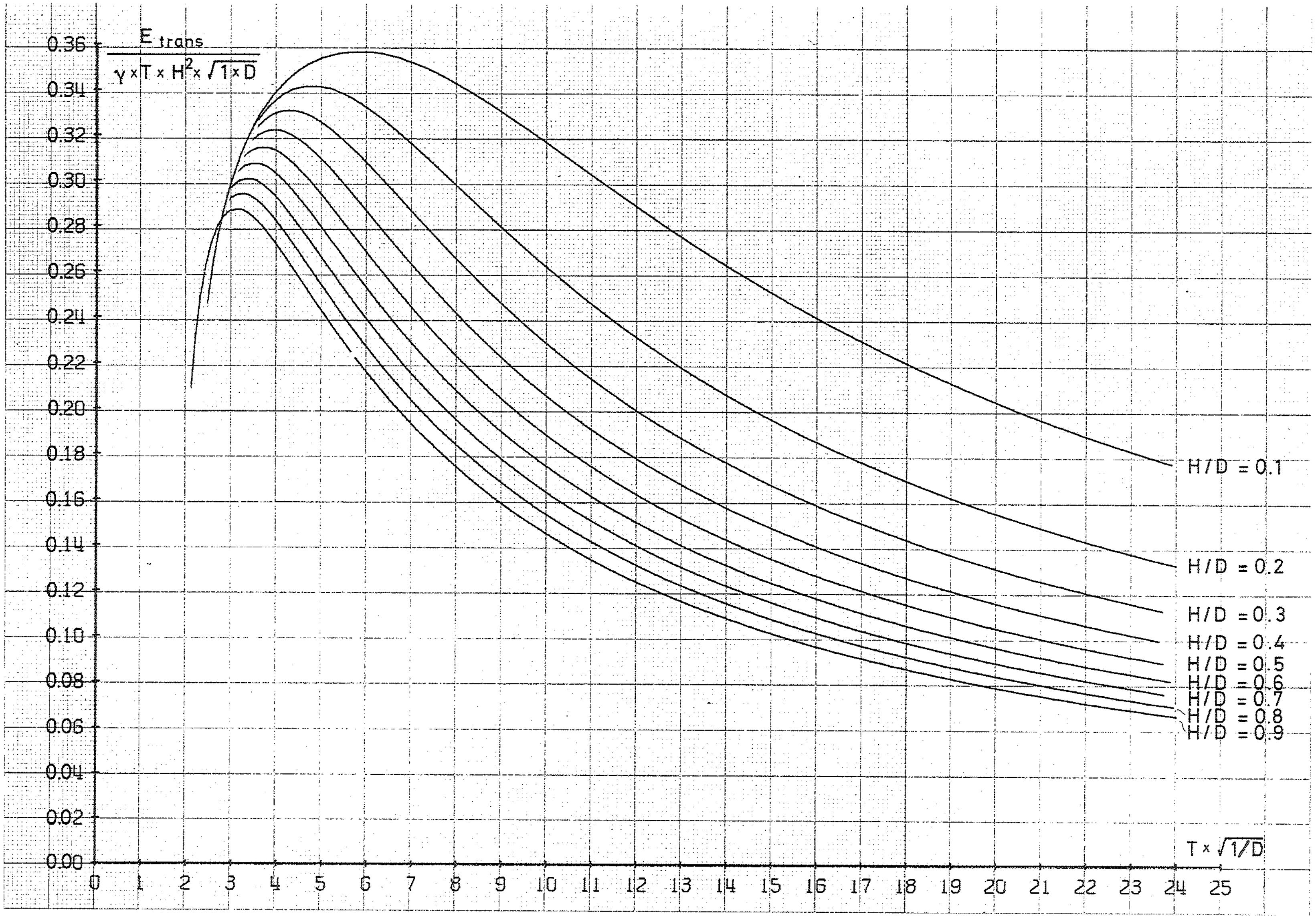
GRAPH 3. An expanded view of part of GRAPH 2.



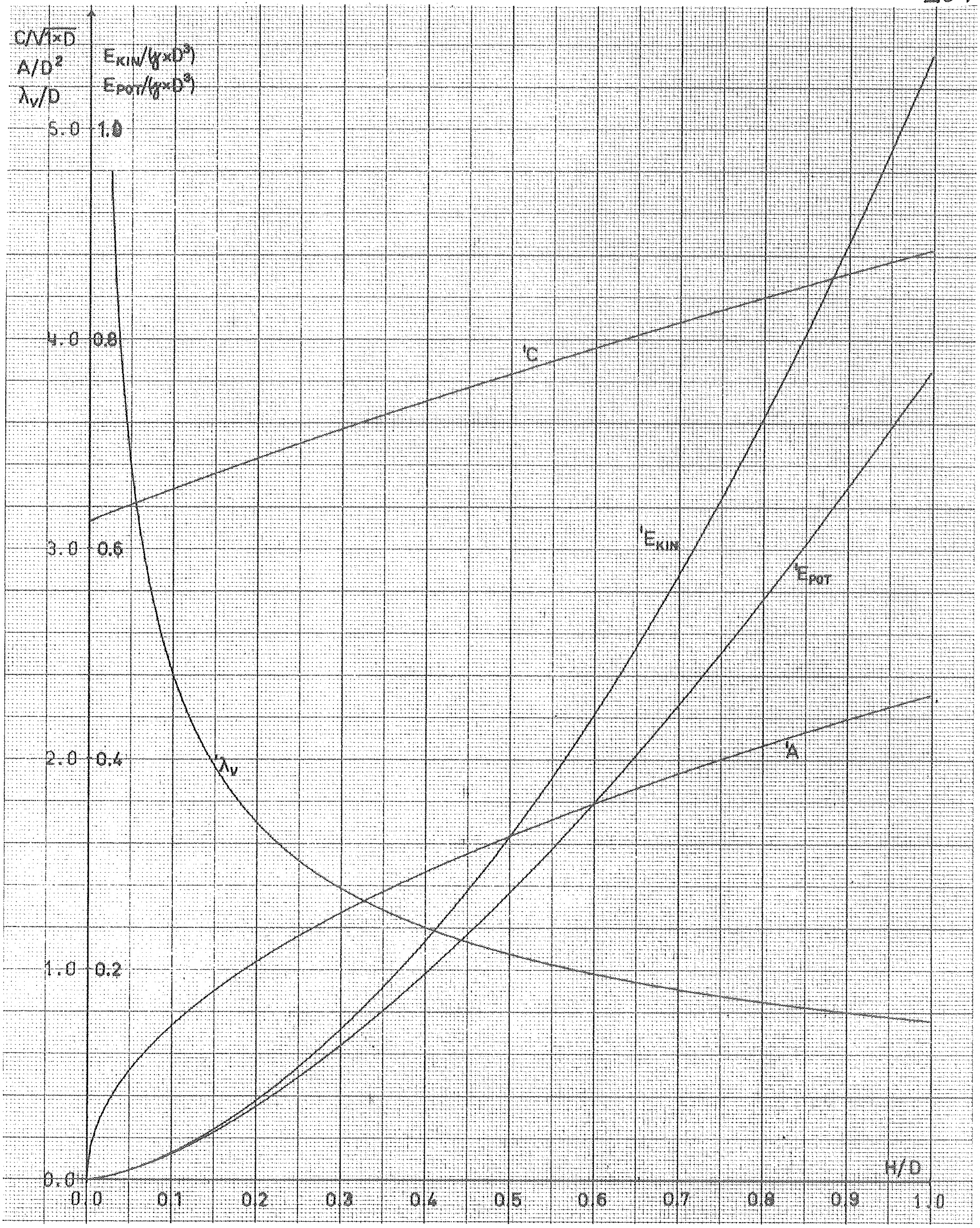
GRAPH 4. Cnoidal wave. As functions of log k_c: wave-length, potential energy, crest-height, crest-volume, distance from the crest to the point of inflection and to the point on mean water level, and K(k).



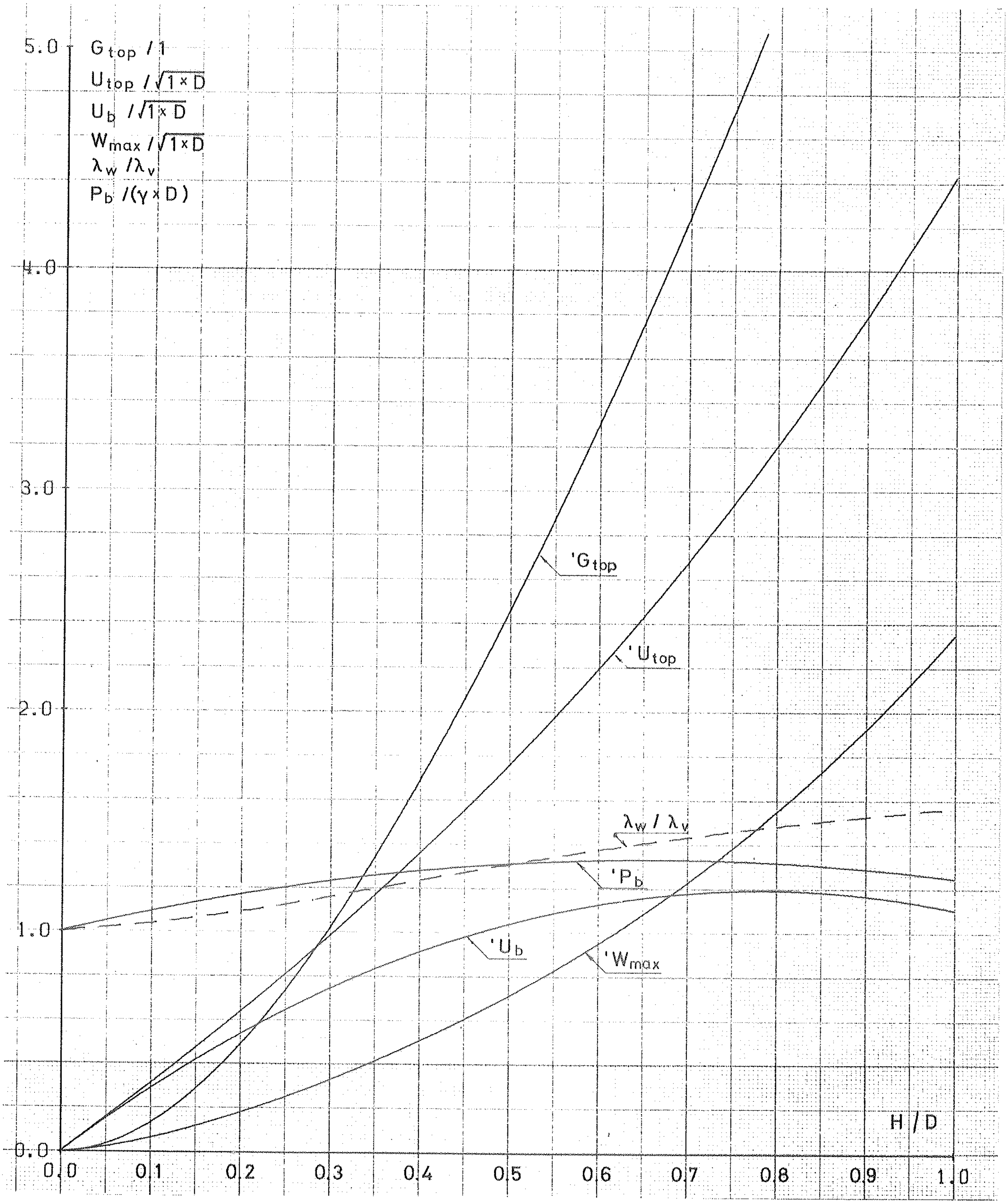
GRAPH 5. Cnoidal wave. The kinetic energy as a function of the wave-period.
(1 is 1 m/sec^2)



GRAPH 6. Cnoidal wave. The transported energy as a function of the wave-period.
(1 is 1 m/sec^2)



GRAPH 7. Solitary wave. As functions of the wave-height and mean water depth: celerity, kinetic and potential energy, wave volume, and distance from the crest to the point of inflection.



GRAPH 8. Solitary wave. As functions of the wave-height and mean water depth: particle velocities and acceleration, pressure, and the distance from the crest to the point of maximum vertical velocity.

NOTATION

a	a constant.
cn	Jacobian elliptic cosine function.
	When $F(\phi,k) = \theta$ then $\text{cn}\ \theta = \cos \phi$.
	For $k = 0$: cn = cos. For $k = 1$: cn = sech.
cosh	hyperbolic cosine.
dn	Jacobian elliptic function. $\text{dn}^2 = k_c^2 + k^2 \cdot \text{cn}^2$
g	acceleration of gravity.
k	modulus in the elliptic functions.
k_c	$= \sqrt{1 - k^2}$, complementary modulus.
ln	natural logarithm, base $e = 2.71828$.
log	common logarithm, base 10.
m, n	positive integers.
p(z)	total unit pressure (above atmospheric pressure) at an arbitrary point, (x, z, t).
$p^+(z)$	wave pressure, = p(z) minus hydrostatic pressure.
sech	$= \frac{1}{\cosh}$
sinh	hyperbolic sine.
sn	Jacobian elliptic sine function.
	When $F(\phi,k) = \theta$ then $\text{sn}\ \theta = \sin \phi$.
	For $k = 0$: sn = sin. For $k = 1$: sn = tanh.
t	time.
tanh	hyperbolic tangent.
u(z)	horizontal particle velocity at an abitrary point (x, z, t).
u_b	u(z) at the bottom, $z = 0$.
u_s	u(z) at the surface, $z = D + \eta$.

$w(z)$ vertical particle velocity at an arbitrary point (x, z, t).

w_s $w(z)$ at the surface, $z = D + \eta$.

x horizontal co-ordinate.

z vertical co-ordinate.

A volume per unit width of the crest, or the water above mean water level.

Arccos the inverse cosine.

$B_1(\eta)$, $B_2(\eta)$, $B_3(\eta)$ constants depending on η.

C wave-celerity.

C_{cnoi} C for cnoidal waves in comparison of theories.

C_{sine} C for sinusoidal waves.

D mean water depth.

$E(k)$ complete elliptic integral of the second kind.

$E(\phi,k)$ $= \int_0^{\phi} \sqrt{1 - k^2 \cdot \sin^2\Theta} \cdot d\Theta$ is the incomplete elliptic integral of the second kind.

$E(\frac{\pi}{2},k) = E(k)$.

E_{kin} (or E_{KIN}), mean value of the kinetic energy. For the cnoidal wave: per horizontal unit area. For the solitary wave: per unit width for the whole wave.

E_{pot} (or E_{POT}), potential energy. Units as for E_{kin}.

E_{trans} (or E_{TRANS}), transported energy, or the energy flux over a wave-period, per unit width.

$E_{flux}(\eta)$ energy flux through a vertical of unit width at η.

$F(\phi, k) = \int_0^{\phi} \frac{d\Theta}{\sqrt{1 - k^2 \cdot \sin^2\Theta}}$, the incomplete elliptic integral of the first kind. $F(\frac{\pi}{2}, k) = K(k)$.

G_{top} — particle acceleration (vertical) at the free surface of the crest $z = D + \eta_c$.

H — wave-height.

I_{fr} — horizontal momentum per unit width of the whole crest.

$K(k)$ — complete elliptic integral of the first kind.

L — wave-length.

L_{sine} — sinusoidal wave-length.

P_b^+ — wave pressure below the crest at the bottom, $z = 0$.

Q — water discharge through a vertical.

R — $= \frac{2 \cdot \pi}{L}$.

T — wave-period.

U — horizontal particle velocity below the crest.

U_b — U at the bottom, $z = 0$.

U_{top} — U at the free surface, $z = D + \eta_c$.

W — $w(z)$ below the point where $W = W_{max}$.

W_{max} — vertical particle velocity, where it is greatest (at the free surface $z = D + \eta$ and λ_W).

γ — unit weight of the water.

η — elevation of the free surface.

η_c — η for the crest, crest height.

η_t — η for the trough, trough height (negative).

η_v — η for the point of inflection.

η_o — $= 0$, η for the point on mean water level.

θ — $= 2 \cdot K(k) \cdot (\frac{x}{L} - \frac{t}{T})$ for the cnoidal wave, and

$= \frac{\sqrt{3}}{2 \cdot D} \cdot \sqrt{\frac{H}{D}} \cdot (x - C \cdot t)$ for the solitary wave.

θ_v	θ for the point of inflection.
θ_o	θ for the point on mean water level.
λ_v	horizontal distance between the crest and the point of inflection.
λ_W	horizontal distance between the crest and the point with W_{max} .
λ_o	horizontal distance between the crest and the point on mean water level.
μ	a variable.
π	$= 3.14159265$.
φ	the velocity potential at an arbitrary point (x, z, t).
φ_o	φ at the bottom $z = 0$.
ϕ	the amplitude in elliptic integrals and functions.
ϕ_v	ϕ for the point of inflection.
ϕ_o	ϕ for the point on mean water level.

$\Delta E_{kin}(\eta)$	kinetic energy per horizontal unit area.
Θ	a variable.
1	in the dimensionless expressions on the graphs 1 means the acceleration 1 metre/(second)^2.

SIMPLIFIED EXPRESSION FOR THE TRANSPORTED ENERGY ?

After the appearance of the expression for the energy flux of this chapter, Svendsen succeeded in finding the more simple expression

$$E_{flux} = \rho g\, c \eta^2 \tag{A1}$$

Svendsen uses a coordinate system that originates at the mean water level instead of at the bottom which leaves out a term as $\gamma c \eta D$ from E_{flux}, the term that will be zero by integration over a period to find the transported energy, E_{trans}. As mentioned, E_{trans} is independent of where we have $z = o$. So eq. A1 is the same as the first and the most important term in the expression for E_{flux} of this chapter, eq. 52.

The transported energy is, using eq. A1,

$$E_{trans} = \int_o^T E_{flux}\, dt = \rho g \int_o^T \eta^2 c\, dt = \rho g \int_o^L \eta^2\, dx \tag{A2}$$

The potential energy is

$$E_{pot} = \frac{1}{2} \gamma \int_o^L \eta^2\, dx \tag{A3}$$

So we see from eq. A1 and A3 that

$$E_{trans} = 2\, E_{pot} \tag{A4}$$

This expression agrees with the classical engineering considerations on shallow water waves: The limit of cnoidal waves is the solitary wave. When the water before and after the passage of the wave is calm and with the same depth, all the energy must accompany the wave. The energy of the wave is partly potential, partly kinetic, so

$$E_{trans} = E_{pot} + E_{kin} \tag{A5}$$

From the first order theory it is known that

$$E_{kin} = E_{pot} \tag{A6}$$

so eq. A5 gives

$$E_{trans} = 2\, E_{pot} \tag{A7}$$

the same as eq. A4.

Eq. A7 has also been used in practice in the need of a better expression for the transported energy of cnoidal waves.

Svendsen's expression for E_{flux} in eq. A1 is seen to be a little different from the expression of this chapter. So it is the question if it is possible in two different ways to reach two different, but correct results. Because then the more simple expression of eq. A1 will be preferable, also because it is exactly the same as given by the sinusoidal theory.

Comparing eq. A1 with eq. 52ª it is seen that Svendsen apparently has neglected terms of third order magnitude. As it is clear to see from e.g. eq. A3, the energy of a first order wave is of second order, and the energy of a second order wave must then include third order terms. So if there during the development of eq. A1 will be terms of third order magnitude they must be included.

So let us consider the development as given by Svendsen in ref [21]. We will use the same notation as there.
The energy flux is defined in eq. 5.1 -3 in [21]

$$E_f = \int_{-h}^{\eta} \rho u \left\{ p^+/\rho + \frac{1}{2} (u^2 + v^2) \right\} dy \tag{A8}$$

where u is given by 3.3 - 16 [21]

$$u = c \frac{\eta}{h} - c \frac{\eta^2}{h^2} + \frac{1}{2} c h \left(\frac{1}{3} - \frac{(y+h)^2}{h^2}\right) \eta_{xx} - c \frac{\bar{\eta}}{h} + O(\varepsilon^6) \tag{A9}$$

Here η is measured from the mean energy level, instead of the mean water level, but with eq. 3.3 - 12 [21]

$$\eta = \eta_M + \bar{\eta} \tag{A1o}$$

and eq. 3.3 - 13 [21] it is seen that u can be written without the term

$$c \frac{\bar{\eta}}{h} \tag{A11}$$

if η is measured from the mean water level, what we then will do.

v has a magnitude so it is without importance in eq. A8.
p^+ is given by eq. 3.6 - 1o [21]

$$p^+ = \rho g \left[\eta + \frac{1}{2} h^2 \left(1 - \frac{(y+h)^2}{h^2}\right)\eta_{xx}\right] + o\,(\varepsilon^6) \tag{A12}$$

Then we get for the energy flux

$$E_f = \rho c \int_{-h}^{\eta} \left[\frac{\eta}{h} - \frac{\eta^2}{h^2} + \frac{1}{2} h \left(\frac{1}{3} - \frac{(y+h)^2}{h^2}\right)\eta_{xx}\right]$$

$$\left\{ g \left[\eta + \frac{1}{2} h^2 \left(1 - \frac{(y+h)^2}{h^2}\right)\eta_{xx}\right] + \frac{1}{2}\left[c^2 \left(\frac{\eta}{h}\right)^2 + \dots\right]\right\} dy \tag{A13}$$

In the last term we use $c^2 = gh$, neglecting higher order terms, so

$$E_f = \rho c \int_{-h}^{\eta} \left\{ g \frac{\eta^2}{h} - g \frac{\eta^3}{h^2} + \frac{1}{2} gh\,\eta \left(\frac{1}{3} - \frac{(y+h)^2}{h^2}\right)\eta_{xx}\right.$$

$$\left. + \frac{1}{2} g\, h\,\eta \left(1 - \frac{(y+h)^2}{h^2}\right)\eta_{xx} + \frac{1}{2} g \frac{\eta^3}{h^2}\right\} dy$$

$$= \rho g c \int_{-h}^{\eta} \left\{ \frac{\eta^2}{h} - \frac{1}{2}\frac{\eta^3}{h^2} + h\,\eta\left(\frac{2}{3} - \frac{(y+h)^2}{h^2}\right)\eta_{xx}\right\} dy$$

$$= \rho g c \left[\eta^2 + \frac{\eta^3}{h} - \frac{1}{2}\frac{\eta^3}{h} + \frac{1}{3} h^2 \eta\,\eta_{xx}\right] \tag{A14}$$

So if we use the same principles as otherwise used in [21] we find for the cnoidal wave, which is one order higher than the sinusoidal wave, that the energy flux of ref. [21] should have been

$$E_f = \rho g c \eta^2 + \rho g c \left[\frac{1}{2}\frac{\eta^3}{h} + \frac{1}{3} h^2 \eta\,\eta_{xx}\right] \tag{A15}$$

If the same principles of neglecting terms and the same reference level is used on eq. 52' it will be exactly the same as eq. A15. There is one disputed term in eq. 52'

$$- \frac{\gamma}{6g} c^3 D^2 \frac{\eta}{(D+\eta)^2} \left(\frac{\partial \eta}{\partial x}\right)^2 \tag{A16}$$

After both the principles of this chapter and of ref [21], this term can be neglected as being of higher order than the included third order terms (after shallow water considerations). But it can never be wrong to include a higher order negligible term as long as the expression for E_{flux} will not be claimed to be of higher order.

But there is a reason not to drop it, if not wanted. It is seen that according to the principles for waves on arbitrary depth the magnitude of eq. A16 is the same as for the terms included (of third order). So the term can as well be included having the sinusoidal limit in mind.

Even though the energy flux in ref [21] is not correct for a cnoidal wave, it could be so lucky that the expression for the transported energy by integration of E_{flux} happened to be correct. One of the forgotten terms in E_{flux} will not contribute to E_{trans}, but the other will.

So the classical expression of eq. A4 is not fully sufficient for a cnoidal wave.

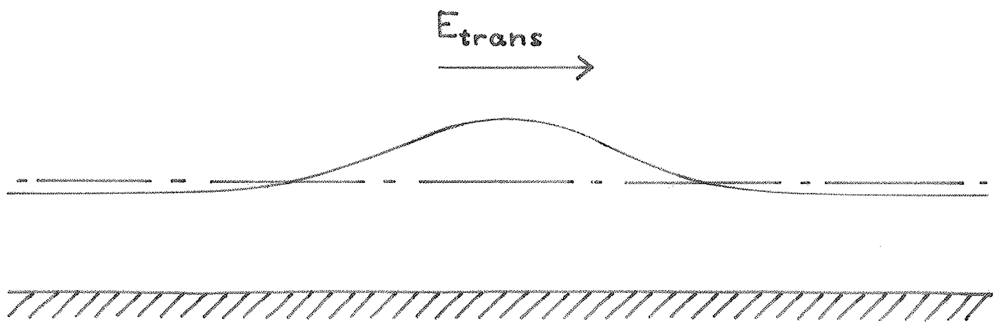


Fig. A1. The transported energy for a shallow water cnoidal wave is not just twice the potential energy.

NUMERICAL EXAMPLE

Let us consider the wave with the period $T = 10$ sec, which on infinite deep water has the wave height $H_o = 3$m. We then find

$$L_o = 156 \text{ m} \quad ; \quad c_o = 15.6 \text{ m/sec}$$

$$E_{trans} = \frac{1}{16} \gamma H_o^2 L_o = \frac{1}{16} \gamma 3^2 \cdot 156 = 88 \gamma$$

We then want to find the wave height H for the same wave when it has reached the depth of $D = 5$ m. We get

$$T \sqrt{1/D} = 10 \sqrt{1/5} = 4.5$$

(1 is 1 m/sec^2). Then we can draw a line through $T \sqrt{1/D} = 4.5$ on graph 6 (and the other graphs) as shown on fig A2.

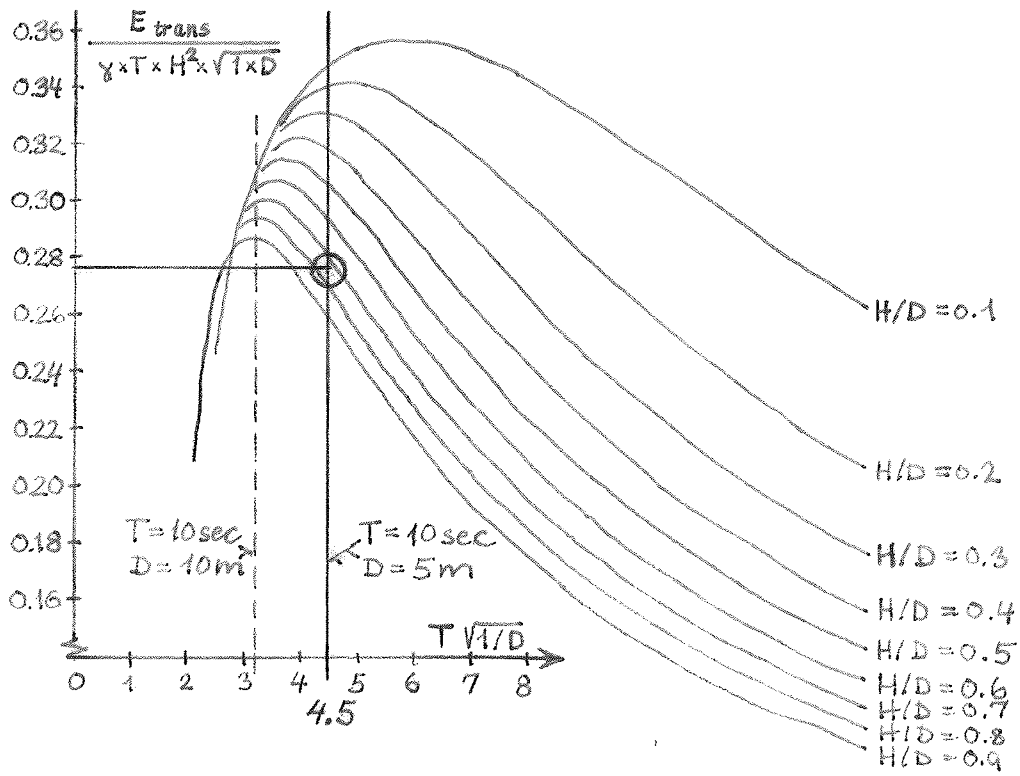


Fig. A2. Determination of the wave height. Part of graph 6.

We get

$$\frac{E_{trans}}{\gamma\ T\ H^2\sqrt{1\cdot D}} = \frac{E_{trans}}{(H/D)^2\ \gamma T D^2\sqrt{1\cdot D}} = \frac{88}{(H/D)^2\ 10.5^2\ \sqrt{5}} = \frac{0.157}{(H/D)^2}$$

By trial and error we then in two steps find

$$H/D = 0.75 \quad \text{for} \quad \frac{E_{trans}}{\gamma\ T\ H^2\sqrt{1\cdot D}} = 0.276$$

This gives $H = 3.7$ m, while the sinusoidal theory would give $H = 3.3$ m.

$T\sqrt{1/D} = 4.5$ and $H/D = 0.75$ is then used on the other graphs, e.g. graph 3 which gives

$$c\ /\ \sqrt{1\cdot D} = 3.68 \quad \text{or} \quad c = 8.2\ \text{m/sec}$$

$$\log k_c = -2.1$$

The wave length is then found from graph 4 and the wave profile from graph 1.

A more relevant depth may be $D = 10$ m, which gives $T\sqrt{1/D} = 3.2$. But then we see that we get outside the region of shallow water cnoidal waves. For higher waves with this D and T it is also not possible to use the Stokes' waves. But then the cnoidal waves of chapter IX can be used.