# CHAPTER VI

# PROGRESSIVE CNOIDAL DEEP WATER WAVES


## ABSTRACTS

The purpose of this chapter is to give the solutions for regular progressive finite amplitude waves on infinite deep water. Different wave solutions of second order are given and the most satisfactory is found to be the cnoidal solution. The wave profile is then described in a way similar to that for the traditional shallow water cnoidal waves, hence the name 'cnoidal'. It is tried to find the solution in a rather simple physical way. The expressions for velocities and pressure are found to be rather simple. Further is shown how rotation can be included.


## INTRODUCTION

In chapter II we considered the first order progressive deep water wave. We then started to neglect higher order terms already for the vertical velocity (eq. 17). We saw though, (eq. 16), that this could mean, that we would have to demand the wave steepness, H/L, rather small for the dropped terms to be negligible. So this time we include higher order terms. Until the wave equation we take all terms along, so that we can assure the quantity of approximations in the final solutions.

This chapter is written so that it in principle can be read independant of chapter II or any other chapters. It could be made only a few lines shorter by connecting it to chapter II, but it is found more convenient for the reader to 'stay within' this chapter. So a few sentences and equations are repeated here.

The idea of why and how to find e.g. cnoidal waves also for deep water is shown in an appendix in an attempt to make the direct developement of the wave more short. It is then understood that a cnoidal solution must exist, although it mathematically may not be so obvious a solution out from considerations of second order sinusoidal solutions.

## BASIC EQUATIONS

We consider a two-dimensional, progressive gravity wave of permanent form on infinite depth, as indicated in fig. 1.

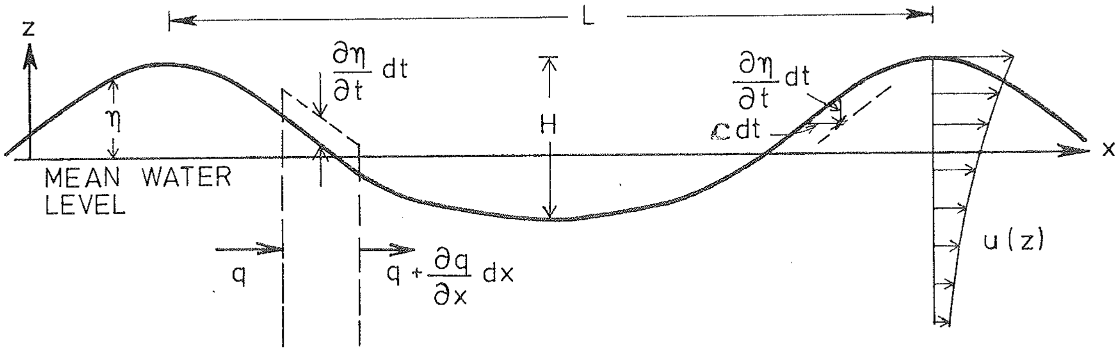



Fig. 1. Definition sketch

The equation of continuity reads

$$\frac{\partial q}{\partial x} = -\frac{\partial \eta}{\partial t} \tag{1}$$

in which q = q(x,t) is the discharge through a vertical, t the time, c the wave celerity and $\eta = \eta(x,t)$ is the surface elevation. As further for a permanent wave

$$-\frac{\partial \eta}{\partial t} = c\frac{\partial \eta}{\partial x} \tag{2}$$

it is seen that for a wave without a resultant discharge

$$q = c\eta \tag{3}$$

To make the following deductions more simple, we assume the vertical distribution of the horizontal particle velocity to be exponential for infinitely deep water, so that u=u(x,z,t)

$$u = qRe^{R(z-\eta)} = c\eta Re^{R(z-\eta)} \tag{4}$$

This can be felt rather as a restrictive assumption. Instead, we might have assumed the distribution to be some unknown function, but, in practice, eq. 4 does not imply any serious restriction, because we might think of u as a series of exponential functions, $u = \Sigma q_i R_i e^{R_i(z-\eta)}$, of which eq.4 is the first term. Finally, it then turns out that the single term in eq. 4 satisfies the final solution of both first and second order with $R = \frac{2\pi}{L}$, L being the wave length. For third order and higher order waves, u will contain several exponential functions with different R values, but this is just what comes out automatically of a perturbation theory based on only eq. 4 with unknown R. The exponential form has, of course, been chosen because it is known to agree with experiments. It is also known that eq. 4 is in accordance with the classical first order potential theory when $R = \frac{2\pi}{L}$.

Using the equation of continuity

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x},$$

and $w = 0$ for $z \to -\infty$, we get the vertical particle velocity, $w = w(x,z,t)$

$$w = c\left[-\frac{\partial \eta}{\partial x} + \eta \frac{\partial \eta}{\partial x} R\right] e^{R(z-\eta)} \tag{5}$$

From eqs. 4 and 5 we get the horizontal particle acceleration $G_x = G_x(x,z,t)$

$$G_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = c^2\left[-\frac{\partial \eta}{\partial x}R + \eta\frac{\partial \eta}{\partial x}R^2\right] e^{R(z-\eta)} \tag{6}$$

The vertical particle acceleration, $G_z = G_z(x,z,t)$, will be

$$G_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} =$$

$$c^2\left\{\left[\frac{\partial^2 \eta}{\partial x^2} - \left[2\left(\frac{\partial \eta}{\partial x}\right)^2 + \eta\frac{\partial^2 \eta}{\partial x^2}\right]R + \eta\left(\frac{\partial \eta}{\partial x}\right)^2 R^2\right] e^{R(z-\eta)}\right.$$

$$\left. + \left[\left[\left(\frac{\partial \eta}{\partial x}\right)^2 - \eta\frac{\partial^2 \eta}{\partial x^2}\right]R + \eta^2\frac{\partial^2 \eta}{\partial x^2}R^2\right] e^{2R(z-\eta)}\right\} \tag{7}$$

Through the vertical dynamic equation for a frictionless fluid

$$-\frac{\partial p}{\partial z} - \rho g = \rho G_z,$$

and $p = 0$ at the surface $z = \eta$, we get an expression for the pressure, $p = p(x,z,t)$. $g$ is the acceleration of gravity and $\rho$ is the density of the water. By differentiation, we derive an expression for $\frac{\partial p}{\partial x}$. Through the horizontal dynamic equation,

$$-\frac{\partial p}{\partial x} = \rho G_x,$$

we obtain an alternative expression for $\frac{\partial p}{\partial x}$. Eliminating $\frac{\partial p}{\partial x}$ from the two equations we get the wave equation

$$\frac{g}{c^2}\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] - \frac{\partial \eta}{\partial x} R\, e^{R(z-\eta)}$$

$$+\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} + \left[-6\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} - \eta\frac{\partial^3 \eta}{\partial x^3} + \left[\left(\frac{\partial \eta}{\partial x}\right)^3 + 2\eta\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right]R\right]\left[1 - e^{R(z-\eta)}\right]$$

$$+\left[\eta\frac{\partial \eta}{\partial x}R^2 - \left[2\left(\frac{\partial \eta}{\partial x}\right)^3 + \eta\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right]R + \eta\left(\frac{\partial \eta}{\partial x}\right)^3 R^2\right] e^{R(z-\eta)}$$

$$+\left[\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} - \eta\frac{\partial^3 \eta}{\partial x^3} + \left[2\eta\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2} + \eta^2\frac{\partial^3 \eta}{\partial x^3}\right]R\right]\frac{1}{2}\left[1 - e^{2R(z-\eta)}\right]$$

$$+\left[\left[\left(\frac{\partial \eta}{\partial x}\right)^3 - \eta\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}\right]R + \eta^2\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial x^2}R^2\right] e^{2R(z-\eta)} = 0 \qquad (8)$$

FIRST ORDER SOLUTION

The terms in eq. 8 are of different order of magnitude, for instance $\frac{\partial \eta}{\partial x}$ and $\frac{\partial^3 \eta}{\partial x^3}$ are small of first order, $\eta\frac{\partial^3 \eta}{\partial x^3}$ of second order etc.

Comparison of the magnitude of the different terms can be made when we have a solution to $\eta$ and given wave parameters, see the appendix.

If we keep only the first order terms and regard the others as negligible, we get the first order wave equation

$$\frac{g}{c^2}\frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3}\frac{1}{R}\left[1 - e^{R(z-\eta)}\right] - \frac{\partial \eta}{\partial x} R\, e^{R(z-\eta)} = 0 \qquad (9)$$

This equation is split into two equations that must be fulfilled simultaneously, an equation of the z-dependent terms

$$\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\,e^{R(z-\eta)} + \frac{\partial\eta}{\partial x}R\,e^{R(z-\eta)} = 0 \tag{10}$$

and an equation of the z-independent terms

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial x^3}\frac{1}{R} = 0 \tag{11}$$

The solution of these two equations is the wellknown first order wave

$$\eta = \eta_1 = \frac{H}{2}\cos k(x-ct)\ ;\quad R = k = \frac{2\pi}{L}\ ;\quad c = \sqrt{\frac{g}{R}} = \sqrt{\frac{gL}{2\pi}} \tag{12}$$

where H is the wave height.

Before we proceed to the higher order solutions of the wave equation, it may be of interest to compare the magnitude of second order terms in eq. 8 with that of first order terms, for instance,

$$-6\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}\bigg/\frac{g}{c^2}\frac{\partial\eta}{\partial x}$$

An approximate evaluation of this ratio can be obtained by insertion of the first order solution, eq. 12, and we find that it becomes

$$6\pi\frac{H}{L}\cos k(x-ct)$$

From this it is seen that H/L must then be unrealistically small for this second order term to be negligible. Hence it would be of interest to develop a more satisfactory theory, as attempted in the following sections.

## SECOND ORDER PERTURBATION SOLUTION

In the wave equation, eq. 8, we now retain the terms of first and second orders and neglect higher order terms. In all second order terms, the first order solution, eq. 12, has been inserted to give us the following second order wave equation:

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\left[1-e^{R(z-\eta)}\right] - \frac{\partial\eta}{\partial x}Re^{R(z-\eta)}$$

$$= 3\left(\frac{H}{2}\right)^2 k^3 \sin 2k(x-ct)\left[1-e^{k(z-\eta)}\right] \tag{13}$$

As for the first order wave equation, we could solve eq. 13 by splitting it into a z-dependent and a z-independent equation.

However, it is an easier procedure to put $z = \eta$ in eq. 13, which gives

$$c = \sqrt{\frac{g}{R}} \tag{14}$$

Then we only need to satisfy the z-dependent equation, which gives $R = k = \frac{2\pi}{L}$, and then we get the solution

$$\eta = \eta_1 + \eta_{2a} = \frac{H}{2}\cos k(x-ct) + \left(\frac{H}{2}\right)^2\frac{k}{2}\cos 2k(x-ct) \tag{15}$$

(In water of finite depth, there would also be a second order term $\eta_{2b}$ with $R = 2k$).

The solution, eq. 15, is reasonable for deep water waves. But in other cases, it is not reasonable to have a solution with a second order wave on top of a first order wave. We will then introduce a solution where the sinusoidal wave has been deformed by a second order correction term in the argument of the cosine function. It is then possible to indicate an alternative solution of the z-dependent equation from eq. 13:

$$\eta = \frac{H}{2}\cos\left(k(x-ct) - \frac{\partial\eta}{\partial x}\right) + \Delta D \tag{16}$$

Within the frames of a second order theory, we can substitute the first order expression for $\frac{\partial\eta}{\partial x}$ in this equation, so that we can write

$$\eta = \frac{H}{2}\cos\theta + \Delta D \; ; \quad \theta = k(x-ct) + \frac{H}{2}k\sin\theta \tag{17}$$

This solution can be shown to be correct also for the third order.

$\Delta D$ is found by integrating eq. 16 and demanding the mean water level to occur at $z = 0$, so for a second (and third) order wave we get

$$\Delta D = \frac{k}{2}\left(\frac{H}{2}\right)^2 = \frac{\pi}{2}\frac{H}{L}\frac{H}{2} \tag{18}$$

A systematic way to find wave solutions like eq. 16 has been suggested by the author (Mejlhede 1975). See also the appendix.

The solutions, eqs. 15, 16, and 17, show that a second order wave can be many things. They all fulfil the wave equation, eq. 8, to the same degree of approximation. But they give different surface profiles, although this difference in the case of deep water is very small.

We then want the solution that comes closest to reality. In the case of shallow water, the cnoidal wave has proved its good value, so we will turn our attention to a similar deep water solution.

## CNOIDAL DEEP WATER WAVE

Another second order solution is the cnoidal wave. In the wave equation, eq. 8, we now consider the third and fourth order terms as negligible.

However, this time we do not substitute the first order solution $\eta_1$ from eq. 12 into the second order terms as we did in deriving eq. 13. We only approximate some of the second order terms in view of the first order solution. For instance, with $\eta = \frac{H}{2}\cos k(x-ct)$, we have

$$\frac{\partial^2\eta}{\partial x^2} = -k^2\eta$$

so that in eq. 8 we can make the substitution

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} = \eta\frac{\partial^3\eta}{\partial x^3} \quad ; \qquad -\eta\frac{\partial\eta}{\partial x}R^2 = \eta\frac{\partial^3\eta}{\partial x^3} = \frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}$$

See the appendix for further explanation.

Eq. 8 will then reduce to

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\left[1-e^{R(z-\eta)}\right] - \frac{\partial\eta}{\partial x}R\,e^{R(z-\eta)}$$
$$+6\left(\frac{2\pi}{L}\right)^2\eta\frac{\partial\eta}{\partial x}\left[1-e^{R(z-\eta)}\right] = 0 \tag{19}$$

As in eq. 14, we get the celerity by substitution of the surface, $z = \eta$

$$c = \sqrt{\frac{g}{R}}$$



After division by $e^{R(z-\eta)}$, the z-dependent equation from eq. 19 is

$$\frac{\partial^3\eta}{\partial x^3} + \frac{\partial\eta}{\partial x}R^2 + 6\left(\frac{2\pi}{L}\right)^3 \eta\frac{\partial\eta}{\partial x} = 0 \tag{20}$$

We shall now show that the solution of this equation is

$$\eta = H\, cn^2 \frac{2K}{L}(x - ct) + \eta_t \tag{21}$$

where $\eta_t$ is the negative trough depth, which, as in the shallow water cnoidal wave theory, is found from the definition of the mean water level by the integration

$$\int_0^L \eta\, dx = 0$$

so that

$$\eta_t = \frac{H}{m}\left(1 - m - \frac{E}{K}\right) \tag{22}$$

K = K(m) and E = E(m) are the complete elliptic integrals of first and second kind, cn is the Jacobian elliptic cosine function, and m is the parameter = the square of the modulus.

For the sinusoidal approximation $m \to 0$, we get $\eta_t \to -\frac{H}{2}$.

The derivatives of $\eta$ : $\partial\eta/\partial x$, $\partial^2\eta/\partial x^2$, and $\partial^3\eta/\partial x^3$ are given in chapter VIII, where also eq. 22 and the sinusoidal limits are given.

By differentiation, eq. 21 gives

$$\frac{\partial^3\eta}{\partial x^3} = -\frac{\partial\eta}{\partial x}\,\frac{16K^2}{L^2}\left[1 - 2m + 3m\, cn^2\frac{2K}{L}(x - ct)\right] \tag{23}$$

This expression is inserted in eq. 20. In the second order term of eq. 20, we substitute $\eta$ by eq. 21 with $\eta_t = -\frac{H}{2}$. We then get

$$-\frac{\partial\eta}{\partial x}\,\frac{16K^2}{L^2}\left[1 - 2m + 3m\, cn^2\frac{2K}{L}(x - ct)\right]$$

$$+\frac{\partial\eta}{\partial x}R^2 + 6\left(\frac{2\pi}{L}\right)^3\frac{\partial\eta}{\partial x} H\left[cn^2\frac{2K}{L}(x - ct) - \frac{1}{2}\right] = 0 \tag{24}$$

This equation is split into two equations, one depending on the cn-terms and the other on the other terms. The cn-dependent equation gives the condition

$$m K^2 = \pi^3 \frac{H}{L} \tag{25}$$

from which m and thereby K(m) are determined by the wave steepness H/L. The cn-independent equation gives with eq. 25

$$R = \frac{4K}{L}\sqrt{1-\frac{m}{2}} \tag{26}$$

For $H/L \to 0$, we get the classical result $R = 2\pi/L$. For H/L near the maximum practical value, R is only a few per cent from $2\pi/L$ (the deviation is small compared to the neglected higher order terms), so that, for most practical purposes, we can use $R = k = 2\pi/L$ corresponding to the sinusoidal second order wave, eq. 15.

ROTATION

So far, we have made no assumptions about the rotation, we have only used the equation of continuity and momentum. The rotation can then be calculated for the solutions we have obtained till now. We then find that the rotation in the first order wave is zero as in the classical first order Airy wave.

The rotation for the second order sinusoidal waves is found by eqs. 4, 5 and 15, 16 or 17 with $R = k = 2\pi/L$ to be

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\left(\frac{H}{2}\right)^2 c\, k^3 \frac{1}{2}\, e^{k(z-\eta)} \tag{27}$$

The rotation for the cnoidal wave is a little more complicated to evaluate, but with approximations we get the same expression as in eq. 27, as a correct second order value.

The rotation in eq. 27 is not in accordance with the classical Stokes' second order theory, which assumes irrotational flow. Hence we would like to be able to change $\Omega$ in eq. 28 to any small arbitrary value.

This can be done by changing the velocity in eq. 4 to

$$u_{rot} = c\eta k e^{k(z-\eta)} + \delta c\left(\frac{H}{2}\right)^2 k^2 e^{k(z-\eta)} + c\left(\frac{H}{2}\right)^2 F(z) \tag{28}$$

The constant $\delta$ and the function F(z) must be chosen so that the last two terms are of second order.

If we want the wave to be without a net flow, we only need to include the proper arbitrary constant in F(z).

The second order rotation in eq. 27 is then changed to

$$\Omega = \left(\frac{H}{2}\right)^2 c k^3 e^{k(z-\eta)} \left[\delta - \frac{1}{2}\right] + c\left(\frac{H}{2}\right)^2 \frac{\partial F(z)}{\partial z} \qquad (29)$$

so that for $\delta = \frac{1}{2}$ and $F(z) = 0$, we get irrotational waves.

When using $u_{rot}$ from eq. 28 instead of u from eq. 4, we see that the second order expressions for w, $G_x$ and $G_z$ in eqs. 5, 6 and 7 are unchanged, so that the second order wave equation in eq. 8 is unchanged. This means that our second order wave solutions are unaffected by second order rotation except for the horizontal particle velocity, which is in agreement with the work of others on rotational waves.

ROTATIONAL WAVES

As an example of how to use the theory in this paper to find waves with arbitrary rotation, we consider a second order wave with a first order rotation. In the appendix, the problem of a second order wave on a specific exponential first order shear flow has been solved. We will here consider a more general shear flow, writing for u

$$u = c\eta R e^{R(z-\eta)} + c\frac{H}{2}k\sum \delta_i n_i e^{n_i k(z-\eta)} \qquad (30)$$

where $\delta_i$ and $n_i$ are freely chosen constants, so that the last term in eq. 30 is kept as a first order quantity. We then get the solution for the surface profile

$$\eta = \eta_1 + \eta_{2a} + \sum(\eta_{2ci} + \eta_{2di}) \qquad (31)$$

where $\eta_1$ and $\eta_{2a}$ are known from eqs. 12 and 15. We get $\eta_{2ci}$ and $\eta_{2di}$ from eqs. 41 and 42 in the appendix

$$\eta_{2ci} = \left(\frac{H}{2}\right)^2 k\delta_i n_i \cos k(x-ct) \qquad \text{with } R = n_i k \qquad (32)$$

$$\eta_{2di} = -\left(\frac{H}{2}\right)^2 k\delta_i \frac{n_i^2}{n_i+2} \cos k(x-ct) \qquad \text{with } R = (n_i+1)k \qquad (33)$$

We get the celerity from eq. 44

$$c = \sqrt{\frac{g}{k}}\left[1 + Hk\sum \frac{\delta_i n_i}{n_i+2}\right] \qquad (34)$$

By selecting the right combinations of $\delta_i n_i$, we can describe most of the practically important rotations of first order and find their effect on a second order wave as far as profile, celerity, kinematics and dynamics are concerned. The same procedure can be used to study the effect of rotation on higher order waves.

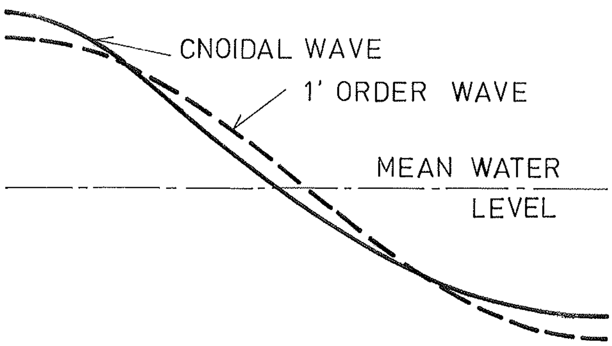



Fig. 2. Cnoidal deep water wave compared to the first order wave for H/L = 10%. The second order waves in eqs. 16 and 17 will give the same profile. The perturbation solution from eq. 15 will be only slightly different.

## APPENDIX I

SOLUTION OF A SECOND ORDER WAVE WITH FIRST ORDER ROTATION

We will here develop a description of a wave propagating on a specific shear flow. We write the horizontal particle velocity as

$$u_1 = c\eta R e^{R(z-\eta)} + c\frac{H}{2} n k e^{nk(z-\eta)}$$

$$= u + c\frac{H}{2} n k e^{nk(z-\eta)} \tag{35}$$

where u is the usual value from eq. 4. The equation of continuity gives $w_1$ as by eq. 5

$$w_1 = w + c\frac{H}{2}\frac{\partial\eta}{\partial x} n k e^{nk(z-\eta)} \tag{36}$$

As before, we get the horizontal particle acceleration

$$G_{x1} = G_x + c^2\frac{H}{2}\frac{\partial\eta}{\partial x}\Big[(nk)^2 e^{nk(z-\eta)}$$

$$+ (R-nk)nk\, e^{(R+nk)(z-\eta)}\Big] \tag{37}$$

and the vertical particle acceleration

$$G_{z1} = G_z - c^2\frac{H}{2}\frac{\partial^2\eta}{\partial x^2} nk\left[e^{nk(z-\eta)} + e^{(R+nk)(z-\eta)}\right] \tag{38}$$

By substitution of the first order solution, eq. 12, into the second order terms we get the complete second order wave equation

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial x^3}\frac{1}{R}\left[1-e^{R(z-\eta)}\right] - \frac{\partial\eta}{\partial x} R e^{R(z-\eta)} =$$

$$3\left(\frac{H}{2}\right)^2 k^3 \sin 2k(x-ct)\left[1-e^{k(z-\eta)}\right]$$

$$+\left(\frac{H}{2}\right)^2 k^3 \sin k(x-ct)\Big\{n^2 e^{nk(z-\eta)}$$

$$+(1-n)n\, e^{(n+1)k(z-\eta)} + \left[1-e^{nk(z-\eta)}\right]$$

$$+\frac{n}{n+1}\left[1-e^{(n+1)k(z-\eta)}\right]\Big\} \tag{39}$$

This equation has a solution of the form

$$\eta = \eta_1 + \eta_{2a} + \eta_{2c} + \eta_{2d} \tag{40}$$

where $\eta_1$ is given by eq. 12 and $\eta_{2a}$ by eq. 15. $\eta_{2c}$ and $\eta_{2d}$ are the solutions to the new terms. Considering the terms with $e^{R(z-\eta)}$ on the left side and $e^{n\,k(z-\eta)}$ on the right side, we must demand $R = nk$ and we get the solution

$$\eta_{2c} = \left(\frac{H}{2}\right)^2 n k \cos k(x-ct) \qquad \text{with} \qquad R = nk \tag{41}$$

In the same way we obtain

$$\eta_{2d} = -\left(\frac{H}{2}\right)^2 \frac{n^2}{n+2} k \cos k(x-ct) \qquad \text{with} \qquad R = (n+1)k \tag{42}$$

Substitution of $\eta_1 + \eta_{2a} + \eta_{2c} + \eta_{2d}$ into the wave equation, eq. 39, gives after some calculation an equation for the determination of the second order celerity

$$\left[\frac{g}{c^2} - k\right] \frac{H}{2} k \sin k(x-ct) = -\left(\frac{H}{2}\right)^2 k^3 \sin k(x-ct) \frac{4n}{n+2} \tag{43}$$

which gives

$$c = \sqrt{\frac{g}{k}} \left[1 + Hk \frac{n}{n+2}\right] \tag{44}$$

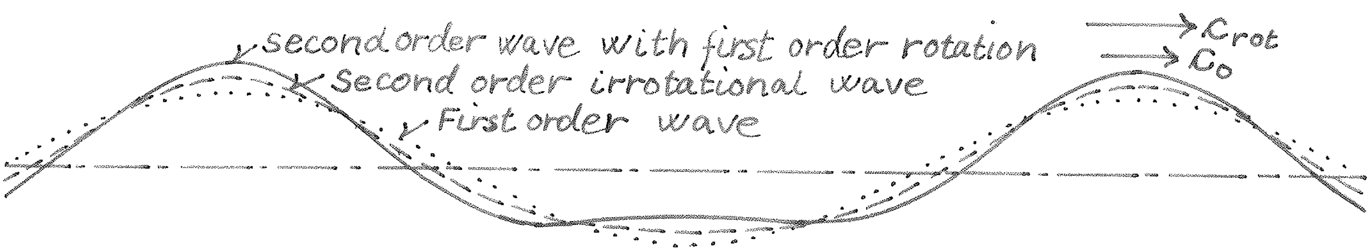



Fig. 3. Second order wave with rotation of first order magnitude (i.e. a wave on a shear stream) has second order changes in the surface profile and the celerity.

THIRD ORDER SINUSOIDAL WAVE

In this appendix we will briefly give the third order sinusoidal solution. It can be found in the same way as the sinusoidal second order solution. In the wave equation, eq. 8, we substitute $\eta_1$, eq. 12, into the third order terms, and $\eta_1$ and $\eta_{2a}$, eq. 15, into the second order terms. We then get a third order wave equation consisting of terms with sin k (x - ct) or sin 3k (x - ct), and with $e^{k(z-\eta)}$ or $e^{2k(z-\eta)}$ or independent of z. From such a wave equation we make minor equations, one depending on $e^{kz}$, one depending on $e^{2kz}$, and one not depending on z.

The final solution, that fulfils all the minor equations, is

$$\eta = \eta_1 + \eta_{2a} + \eta_{3a} + \eta_{3c} \tag{45}$$

where $\eta_{2a}$ is from eq. 15, and $\eta_1$, $\eta_{3a}$, and $\eta_{3c}$ are

$$\eta_1 = \frac{H_1}{2} \cos k(x-ct) \tag{46}$$

$$\eta_{3a} = \left(\frac{H}{2}\right)^3 \frac{3}{8} k^2 \cos 3k(x-ct) \quad \text{with } R = k = \frac{2\pi}{L} \tag{47}$$

$$\eta_{3c\,rot} = \left(\frac{H}{2}\right)^3 k^2 \left[\frac{1}{6} - \frac{\delta}{3}\right] \cos k(x-ct) \qquad \text{with } R = 2k \tag{48}$$

For the wave height, the last three equations give

$$\frac{H_1}{2} = \frac{H}{2} - \left(\frac{H}{2}\right)^3 k^2 \left(\frac{13}{24} - \frac{\delta}{3}\right) \tag{49}$$

The celerity will be

$$c_{rot}^2 = \frac{g}{k}\left[1 + \pi^2\left(\frac{H}{L}\right)^2\left(\frac{1}{3} + \frac{4}{3}\delta\right)\right] \tag{50}$$

In the equations here we have included $\delta$ like in eq. 28, so that the horizontal particle velocity is given as

$$u_{rot} = c\,\eta\,R\,e^{R(z-\eta)} + \delta c \left(\frac{H}{2}\right)^2 k^2 e^{k(z-\eta)} \tag{51}$$

By this we can change the rotation within second order magnitude, so that we for $\delta = \frac{1}{2}$ have irrotational waves, like described by eq. 29. The irrotational third order celerity will then be

$$c^2_{rot=0} = \frac{g}{k}\left[1 + \pi^2\left(\frac{H}{L}\right)^2\right] \tag{52}$$

We also see that rotation of this type will make no changes in the third order surface profile. Rotation will just 'move' part of the wave from $\eta_{3c}$ (eq. 48) to $\eta_1$ (eq. 46), and they are both described by cos k(x - ct). It gives a little change in the distribution of the horizontal velocity, though, through the differences in R.

With different R values for different $\eta$ solutions the full expression for u in eq. 51 will be

$$u_{rot} = c(\eta_1 + \eta_{2a} + \eta_{3a})\, k\, e^{k(z-\eta)} + c\,\eta_{3c}\cdot 2k\, e^{2k(z-\eta)} + \delta c\left(\frac{H}{2}\right)^2 k^2 e^{k(z-\eta)} \tag{53}$$

and correspondingly for w and p.

The solutions of this appendix can then be tested in the wave equation, eq. 8, and seen to be a correct third order solution.

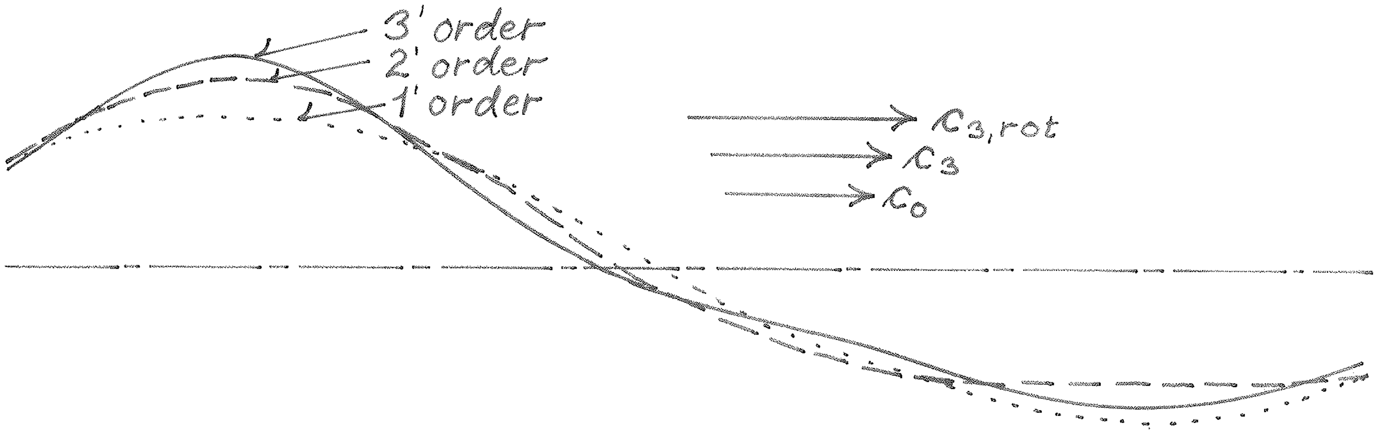



Fig. 4. The celerity of third order depends on the wave height and on the rotation of second order.

EXAMPLES OF ROTATIONAL WAVES

In eq. 30 we assume we have a shear stream of

$$U = c\frac{H}{2}k\,e^{k(z-\eta)} \tag{54}$$

This means that $\delta = 1$ and $n = 1$. Eqs. 31, 32, and 33 will then give the surface elevation

$$\eta = \eta_1 + \eta_{2a} + \eta_{2c} + \eta_{2d} \tag{55}$$

$$\eta_{2c} = \left(\frac{H}{2}\right)^2 k \cos k(x-ct) \qquad \text{with} \quad R = k \tag{56}$$

$$\eta_{2d} = -\left(\frac{H}{2}\right)^2 k \frac{1}{3} \cos k(x-ct) \qquad \text{with} \quad R = 2k \tag{57}$$

The celerity, eq. 34, will be

$$c = \sqrt{\frac{g}{k}}\left[1 + Hk\frac{1}{3}\right] \tag{58}$$

We will compare the celerity with the stream at the surface, $z = \eta$

$$U_s = \frac{1}{2}cHk \tag{59}$$

$$c = c_0 + \frac{1}{3}c_0 Hk \simeq c_0 + \frac{1}{3}cHk \tag{60}$$

$\eta_{2c}$ and $\eta_{2d}$ are of the same type as $\eta_1$ and can as well be said to be included in $\eta_1$. But $\eta_{2d}$ with $R = 2k$ shows us that the velocity distribution in the wave motion is slightly affected by the shear flow.

We now choose $\delta = \frac{1}{2}$ and $n = 2$. We then get for U

$$U = c\frac{H}{2}k\,e^{2k(z-\eta)} \tag{61}$$

so that the flow now will vanish faster with the depth. At the surface we get the same value as before in eq. 59

$$U_s = \frac{1}{2}cHk \tag{62}$$

The celerity in eq. 34 will be

$$c = \sqrt{\frac{g}{k}}\left[1 + Hk\frac{1}{4}\right] \simeq c_0 + \frac{1}{4}cHk \tag{63}$$

Eqs. 32 and 33 will be

$$\eta_{2c} = \left(\frac{H}{2}\right)^2 k \cos k(x - ct) \qquad \text{with} \quad R = 2k \tag{64}$$

$$\eta_{2d} = -\left(\frac{H}{2}\right)^2 k \frac{1}{2} \cos k(x - ct) \qquad \text{with} \quad R = 3k \tag{65}$$

If we consider a wave with the period T = 10 seconds and the wave height H = 3 meters we get L = 156 m, $c_0$ = 15,6 m/sec, H/L = 2%. The stream at the surface chosen in eqs. 59 and 62 will then be close to 1 m/sec.

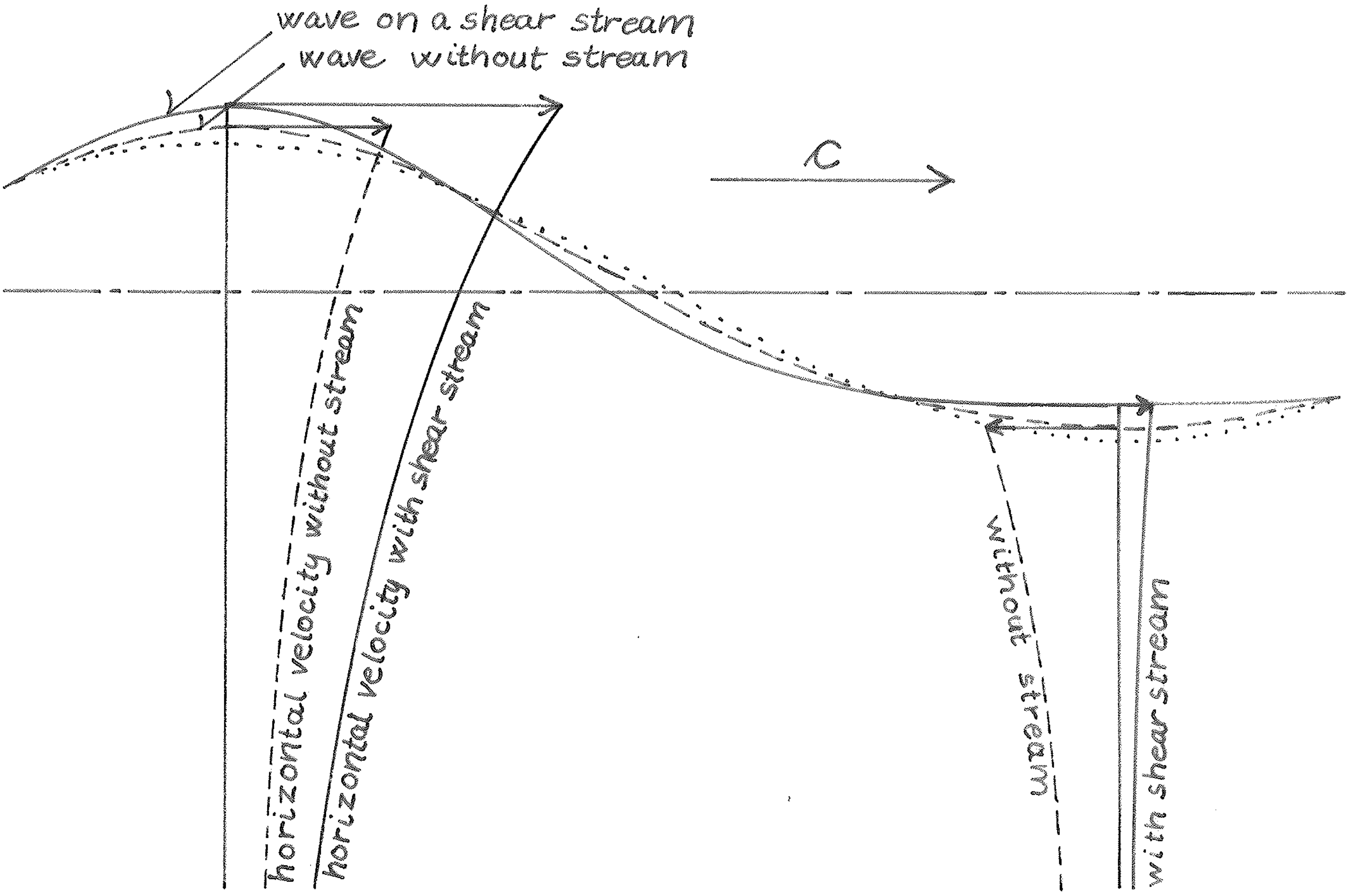



Fig. 5. Second order deep water wave propagating on a shear stream.

## PRESSURE

The particle velocities, u and w, are determined by the equations which were found at the beginning, eqs. 4 and 5, together with the solutions for $\eta$ of first or second order.

In the second order term of w it is hydrodynamically permissible to use first order expressions for $\eta$ and its derivatives. The expression for the pressure p was left out this time so it will be considered here.

Using eq. 7 and the vertical dynamic equation we find to the second order by integration, with R = k

$$\frac{p}{\gamma} = \eta - z + \frac{c^2}{g}\left\{\left[\frac{\partial^2\eta}{\partial x^2}\frac{1}{k} - 2\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\left[1-e^{k(z-\eta)}\right] + \frac{1}{2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\left[1-e^{2k(z-\eta)}\right]\right\} \tag{66}$$

As a boundary condition we used p = o at the surface, z = $\eta$.

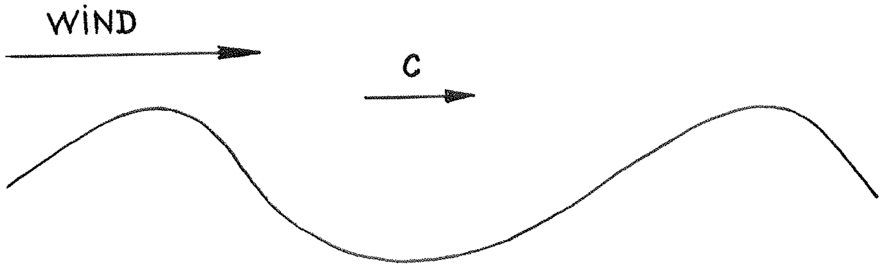



Fig. 6. During the generation of waves by the wind the pressure on the surface will not be p = o. The result will be the asymmetrical wave.

We could also have used a different value. During the generation of waves by the wind we will not have p = o for z = $\eta$. The wind will exert a pressure on the surface by which energy is moved from the wind to the water. The pressure will be negative abovethe crest (compared to atmospheric pressure) and positive above the trough, and because of friction the pressure will be bigger on the back than on the front of the wave crest. If such a pressure distribution on the surface is included in the expression for p, the wave equation, eq. 8 will be changed a little and the result will be an asymmetrical wave. In view of the solution in eq. 16 we can give the solution as

$$\eta = \frac{H}{2}\cos\left(k(x-ct) - \alpha\frac{\partial\eta}{\partial x} - \beta k\eta\right) \tag{67}$$

where then the parameters α and β are determined by the pressure on the surface.

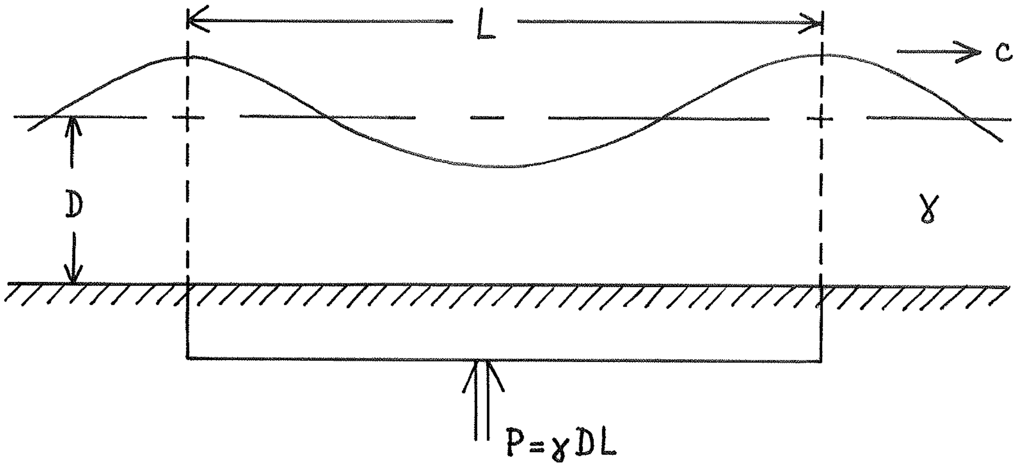



Fig. 7. At the bottom there is a dynamic boundary condition, which is as simple as shown above for the progressive wave. For progressive deep water waves it is special simple.

The pressure in eq. 66 can be made more simple. In all second order terms we can use first order expressions, if we want. But the whole expression can be changed. Eq. 66 is a second order expression, this means that we are allowed to make any third order changes. Such changes can be made with the only purpose of simplifying expressions, but they can also be made with a physical purpose. In eq. 66 we wanted the boundary condition p = o at the surface z = $\eta$ to be fulfilled exactly. Any third order changes must not change this.

The wave pressure at infinite depth is called $p_b^+$, so that

$$\frac{p_b^+}{\gamma} = \frac{p}{\gamma} + z \qquad \text{for} \qquad z \rightarrow -\infty \tag{68}$$

At great depth there is no motion of the water so we will then find that $p_b^+$ must be constant in horizontal direction. Otherwise the difference in pressure would create motion, a horizontal velocity $u_b$, (through the equation of momentum). And we cannot have any motion, because however small $u_b$ is, $u_b$ will by integration over the infinite vertical give an infinite discharge, in disagreement with the equation of continuity in eq. 1. By considering the vertical movement of the center of mass of the water within two verticals one wave length apart, and infinite high (it does not move), we find that at infinite depth we simply have the hydrostatic pressure, so

$$p_b^+ = 0 \tag{69}$$

for a progressive wave.

Eq. 69, which is obvious for the engineer, can be used to make the third order changes of eq. 66. For $z \to -\infty$ we have $e^{k(z-\eta)} \to 0$, so

$$0 = \frac{p_b^+}{\gamma} = \frac{p}{\gamma} + z = \eta + \frac{c^2}{g}\left\{\left[\frac{\partial^2\eta}{\partial x^2}\frac{1}{k} - 2\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right] + \frac{1}{2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\right\} \tag{70}$$

This is used to find a different expression for the coefficient of $[1 - e^{k(z-\eta)}]$ in eq. 66

$$\frac{p}{\gamma} = \eta - z - \left[\eta + \frac{1}{2}\frac{c^2}{g}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\right]\left[1 - e^{k(z-\eta)}\right] + \frac{1}{2}\frac{c^2}{g}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\left[1 - e^{2k(z-\eta)}\right] \tag{71}$$

This is with eq. 12 reduced to

$$\frac{p}{\gamma} = -z + \eta e^{k(z-\eta)} + \frac{1}{2k}\left[\left(\frac{\partial\eta}{\partial x}\right)^2 - \eta\frac{\partial^2\eta}{\partial x^2}\right]\left[e^{k(z-\eta)} - e^{2k(z-\eta)}\right] \tag{72}$$

The last term is of second order (and in practice it is even a small second order term), so we substitute the first order solution of eq. 12 into this term and get

$$\frac{p}{\gamma} + z = \eta e^{k(z-\eta)} + \frac{\pi}{4} H \frac{H}{L} \left[ e^{k(z-\eta)} - e^{2k(z-\eta)} \right] \tag{73}$$

a surprising simple second order expression, for e.g. the cnoidal wave. We just assure that we still have p = o at the surface, $z = \eta$.

Eq. 73 could of course also be found in the mathematical way from eq. 66 using the cnoidal solution of eq. 21 etc., and using usual mathematical approximations, but it is believed that the engineer prefers the more physical considerations of this appendix.

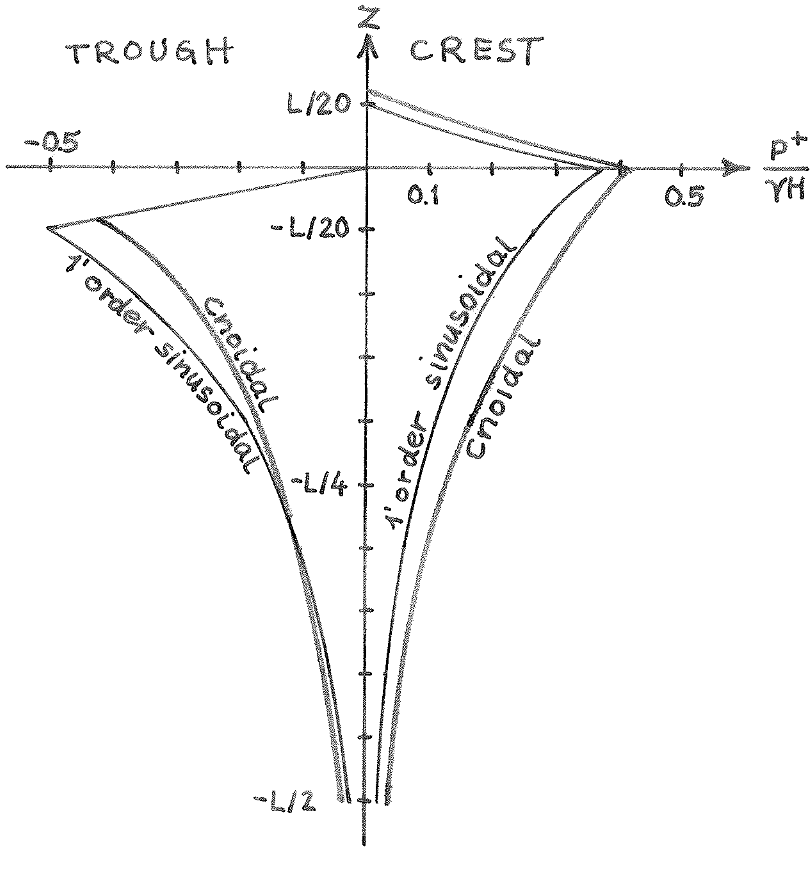



Fig. 8. Comparison of the pressure in the cnoidal wave with the pressure in the first order sinusoidal wave for a progressive deep water wave with the steepness H/L = 1o %.

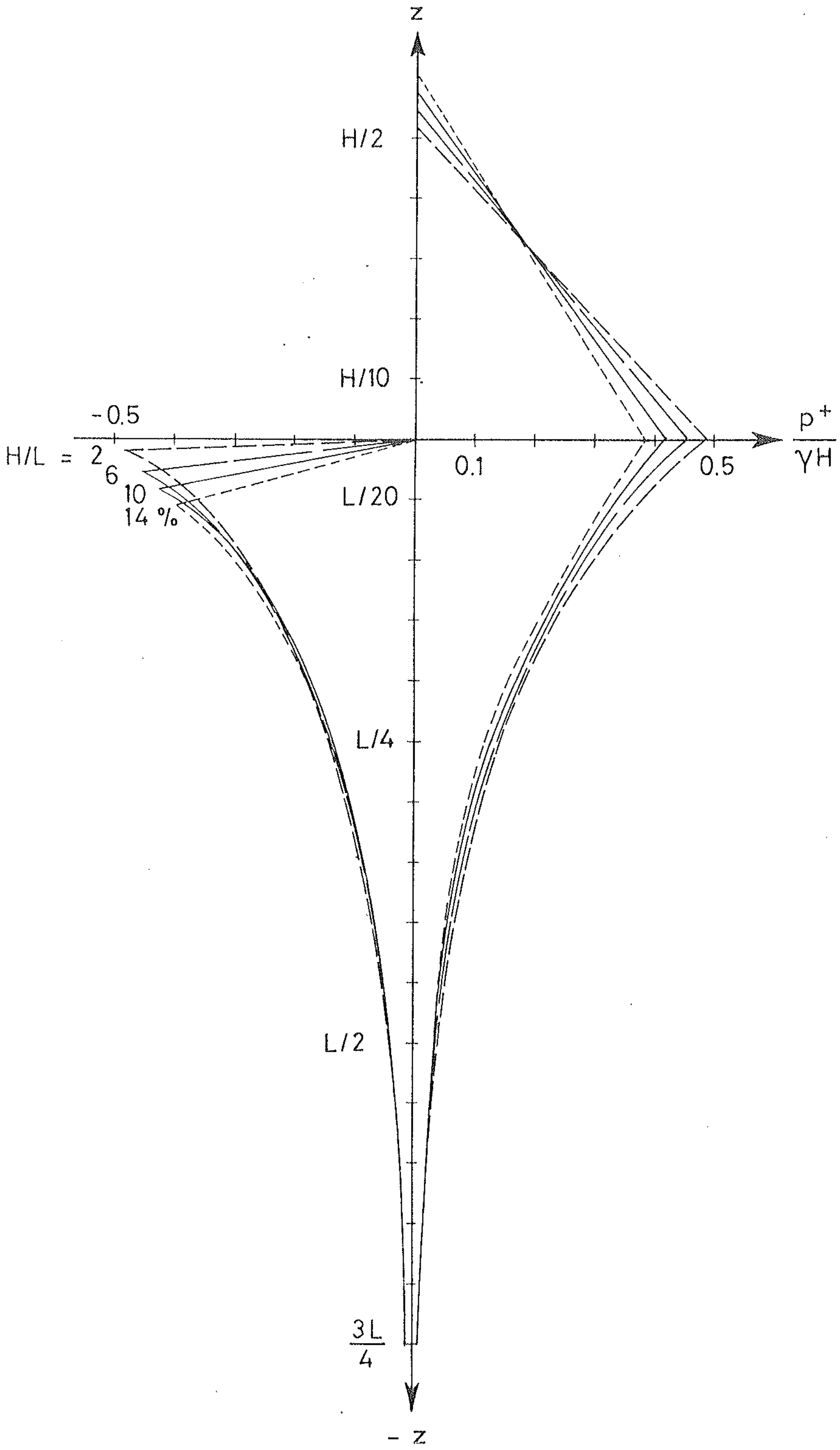



Fig. 9. Maximum positive and maximum negative wave pressure below the progressive cnoidal wave for different steepnesses: H/L = 2%, 6%, 10% and 14%. Note that the unit is different below and above the mean water level, z = 0.

# APPENDIX V

## ORDER OF MAGNITUDE OF TERMS IN THE WAVE EQUATION

Let us consider the wave equation, eq. 8, where all terms are included this time. The terms are not written in a non-dimensional form, but this can be obtained by dividing over by R or multiplying by L. The magnitude of the terms can be found by insertion of the solutions eqs. 12, 15, 16, 17 or 21. It is then found that e.g. $\partial\eta/\partial x$ is proportional to H/L so that $\partial\eta/\partial x$ is of the same magnitude as H/L which is written $\partial\eta/\partial x$ is $o(\frac{H}{L})$. In this way we find for the terms of eq. 8

$\frac{\partial\eta}{\partial x}$ and $\frac{\partial^3\eta}{\partial x^3}\frac{1}{R^2}$ are $o(\frac{H}{L})$

$\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}\frac{1}{R}$ and $\eta\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}$ and $\eta\frac{\partial\eta}{\partial x}R$ are $o(\frac{H}{L})^2$

$(\frac{\partial\eta}{\partial x})^3$ and $\eta\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}$ and $\eta^2\frac{\partial^3\eta}{\partial x^3}$ are $o(\frac{H}{L})^3$

$\eta(\frac{\partial\eta}{\partial x})^3R$ and $\eta^2\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}R$ are $o(\frac{H}{L})^4$

So for the wave steepness, H/L, sufficient small it is correct to neglect the third and fourth order terms in a second order theory, and in a first order theory also to neglect the second order terms. But there are several more terms of second order than of first order so it can be of interest to see how much is neglected for realistic waves.

Let us consider the z-independent equation

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x}+\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}-\frac{9}{2}\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\frac{3}{2}\eta\frac{\partial^3\eta}{\partial x^3}+\left(\frac{\partial\eta}{\partial x}\right)^3R$$
$$+3\eta\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}R+\frac{1}{2}\eta^2\frac{\partial^3\eta}{\partial x^3}R=0 \qquad (74)$$

For all solutions we have $g/c^2 = R$ (eq. 14). Making substitutions like we did just before eq. 19 the two second order terms can be combined to

$$-\frac{9}{2}\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\frac{3}{2}\eta\frac{\partial^3\eta}{\partial x^3}=6\frac{\partial\eta}{\partial x}\eta R^2 \qquad (75)$$

Comparing this with the first term we get

$$R\frac{\partial\eta}{\partial x}+6\frac{\partial\eta}{\partial x}\eta R^2 = R\frac{\partial\eta}{\partial x}(1+6\eta R) \tag{76}$$

With R = 2π/L we get

$$6\eta R = 12\pi\frac{\eta}{L} < 25\frac{H}{L} \tag{77}$$

where we used that for deep water waves $\eta$ will not exceed 0.65 H. In a first order theory we would like to neglect $6\eta R$ compared to 1. But for

$$6\eta R \ll 1$$

we must demand

$$\frac{H}{L} \ll 0.04 \tag{78}$$

and this may not be felt so realistic. In fact, for common waves the second order terms may be even bigger than the first order term.

The first order theory will though not be as bad as it may seem from these considerations. This is due to the control we keep of the resulting expressions. The surface profile is kept between the crest elevation and the trough elevation. The horizontal velocity must by integration over a vertical give the discharge. The vertical velocity must be right at the surface and vanish at infinite depth. The wave pressure must be zero at the surface and vanish at infinite depth. The same type of control should be used on the second order theory. This is done by the cnoidal wave profile.

We will then compare with the rest of the terms of eq. 74. We make the same type of substitutions as before, by which $(\partial\eta/\partial x)^2$ will be

$$\left(\frac{\partial\eta}{\partial x}\right)^2 = \left(\frac{H}{2}\right)^2 R^2 - \eta^2 R \tag{79}$$

Then eq. 74 will be

$$\frac{g}{c^2}\frac{\partial\eta}{\partial x}+6\frac{\partial\eta}{\partial x}\eta R^2+\left(\frac{\partial\eta}{\partial x}\right)^3 R-3\eta^2\frac{\partial\eta}{\partial x}R^3-\frac{1}{2}\eta^2\frac{\partial\eta}{\partial x}R^3=0$$

or

$$R\frac{\partial\eta}{\partial x}\left[\frac{1}{R}\frac{g}{c^2}-6\eta R+\left(\frac{H}{2}\right)^2 R^2-\frac{9}{2}\eta^2 R^2\right]=0 \tag{80}$$

The terms $\frac{1}{R}\frac{g}{c^2}$ and $(\frac{H}{2})^2 R^2$ are combined to give the third order celerity of eq. 5o. We then get for the above equation

$$R\frac{\partial\eta}{\partial x}\left[1+6\eta R\left(1-\frac{3}{4}\eta R\right)\right]=0 \tag{81}$$

In a second order theory we neglect only the last term. So we will want

$$\frac{3}{4}\eta R \ll 1 \tag{82}$$

For the maximum wave steepness, H/L = o.14 we get

$$\frac{3}{4}\eta R < 3\frac{H}{L} < 0.42 < 1 \tag{83}$$

So the second order theory can in the mathematical sense be a good approximation.

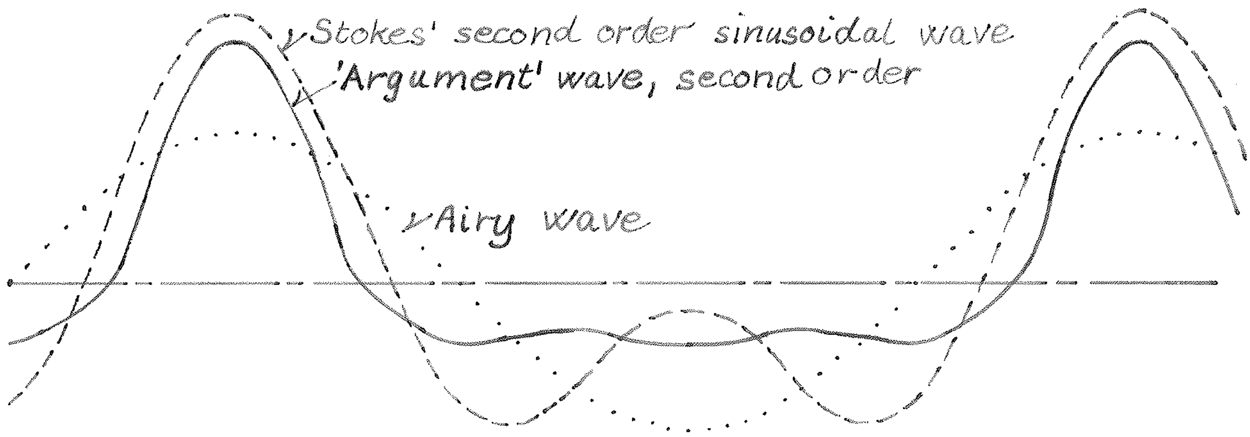



Fig. 10 When the argument wave, eq. 17, is used far beyond its limit (in practice not possible for deep water, but possible in chapter VII), several small 'secondary crests' may appear, while the Stokes' theory only give one 'secondary crest' in the middle of the trough. Hydrodynamically the argument theory is just as good a second order sinusoidal theory as the Stokes' theory, so this shows that there is nothing physical significant about the Stokes' second order 'crest in the trough'. It will therefore be preferred to have a wave theory that avoids the secondary waves at all, that gives a more 'smooth profile', i.e. the cnoidal wave.

SECOND ORDER SUBSTITUTIONS IN THE CNOIDAL WAVE EQUATION

We will now consider the substitutions just above eq. 19 which we made to prepare the wave equation for the cnoidal solution. We will here show, that the teory is mathematical consistent, i.e. we have not dropped any terms of second order magnitude, but neglect terms of third and higher orders when needed. We will show that the substitutions above eq. 19 are correct when neglecting third order terms, and employing the cnoidal solution, eq. 21, for $\eta$.

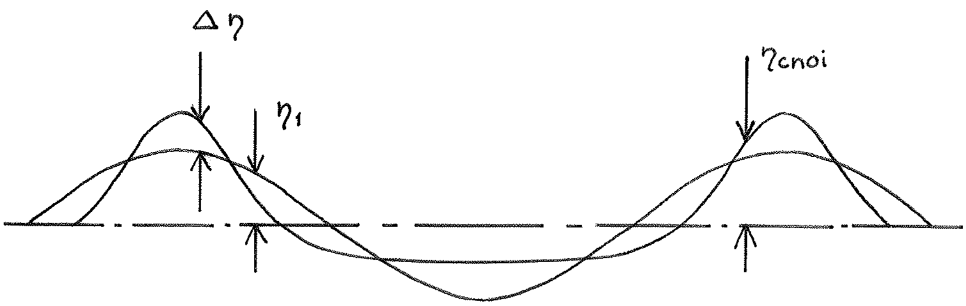



Fig. 11. The cnoidal wave is during the wave development considered to be only a magnitude of second order different from the first order wave.

The second order sinusoidal wave, eq. 15, is

$$\eta = \eta_1 + \eta_{2a} \tag{84}$$

so that $\eta(o(H/L))$ is a sum of the first order wave $\eta_1(o(H/L))$, eq. 12, and a small second order correction term $\eta_{2a}(o(H/L)^2)$.

The second order cnoidal wave could in the same way be written

$$\eta = \eta_{cnoi} = \eta_1 + \Delta\eta \tag{85}$$

where then $\Delta\eta$ $(o(H/L)^2)$ is the small second order difference between the first order wave $\eta_1(o(H/L))$ and the cnoidal wave $\eta_{cnoi}$ $(o(H/L))$. In eq. 84, $\eta_1$ and $\eta_{2a}$ were found separately and then added to give $\eta$. In eq. 85, $\eta_{cnoi}$ was found directly without concern for the difference $\Delta\eta$.

Eq. 85 gives, using eq. 12,

$$\frac{\partial^2\eta}{\partial x^2} = \frac{\partial^2\eta_1}{\partial x^2} + \frac{\partial^2\Delta\eta}{\partial x^2} = -k^2\eta_1 + \frac{\partial^2\Delta\eta}{\partial x^2}$$

$$= k\left(-k\eta + k\Delta\eta + \frac{1}{k}\frac{\partial^2\Delta\eta}{\partial x^2}\right) \tag{86}$$

where $k\eta$ is $o(H/L)$, and where $k\Delta\eta$ and $\frac{1}{k}\frac{\partial^2\Delta\eta}{\partial x^2}$ both are $o(H/L)^2$ and

$$\frac{\partial^3\eta}{\partial x^3} = \frac{\partial^3\eta_1}{\partial x^3} + \frac{\partial^3\Delta\eta}{\partial x^3} = -k^2\frac{\partial\eta_1}{\partial x} + \frac{\partial^3\Delta\eta}{\partial x^3}$$

$$= k^2\left(-\frac{\partial\eta}{\partial x} + \frac{\partial\Delta\eta}{\partial x} + \frac{1}{k^2}\frac{\partial^3\Delta\eta}{\partial x^3}\right) \tag{87}$$

where $\frac{\partial\eta}{\partial x}$ is $o(H/L)$ and where $\frac{\partial\Delta\eta}{\partial x}$ and $\frac{1}{k^2}\frac{\partial^3\Delta\eta}{\partial x^3}$ are both $o(H/L)^2$.

Eqs. 86 and 87 combine to give

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} = \frac{\partial\eta}{\partial x}k\left(-k\eta + k\Delta\eta + \frac{1}{k}\frac{\partial^2\Delta\eta}{\partial x^2}\right)$$

$$= \left(-\frac{\partial^3\eta}{\partial x^3} + k^2\frac{\partial\Delta\eta}{\partial x} + \frac{\partial^3\Delta\eta}{\partial x^3}\right)\left(-\eta + \Delta\eta + \frac{1}{k^2}\frac{\partial^2\Delta\eta}{\partial x^2}\right)$$

$$= \eta\frac{\partial^3\eta}{\partial x^3} + o\left(\frac{H}{L}\right)^3 + o\left(\frac{H}{L}\right)^4 \tag{88}$$

So, in a second order theory (as the cnoidal wave of this paper) the substitution

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2} = \eta\frac{\partial^3\eta}{\partial x^3} \tag{89}$$

is mathematically consistent. The other substitution made in eq. 19 can be considered in the same way.

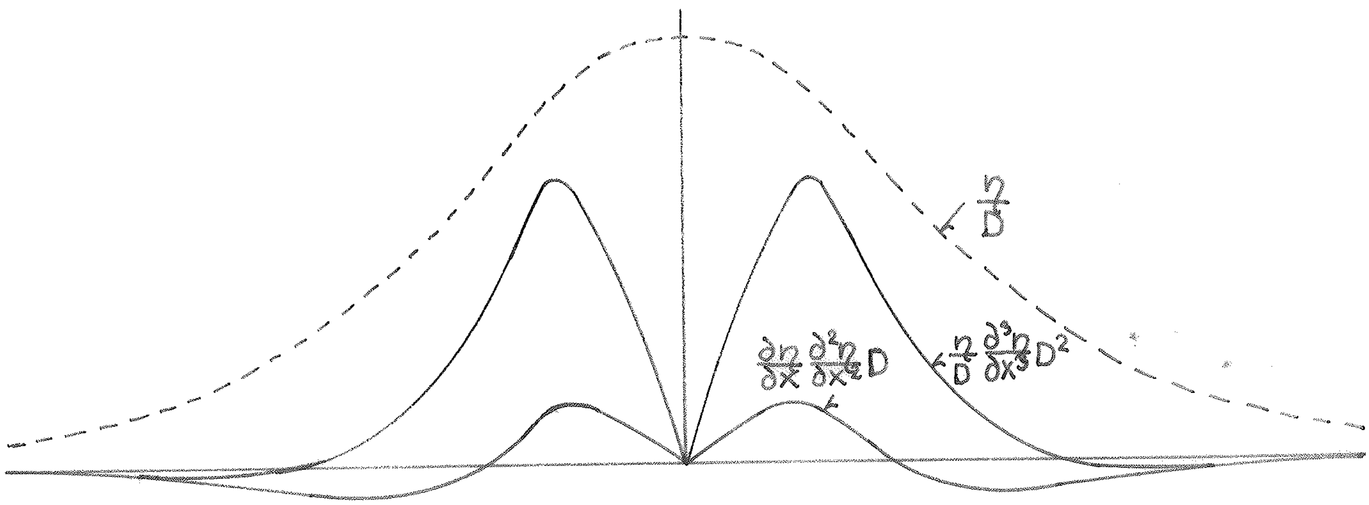



Fig. 12. Comparison of the discussed second order terms for the very extreme case of a solitary wave, H/D = 0.6 (which is highly relevant for chapter IX). The difference may here seem rather big, but the alternative of a (traditional) sinusoidal substitution or neglection is far less attractive. It should also be kept in mind that this substitution is only made for one out of several second order terms in the wave equation.

# APPENDIX VII

## SECOND ORDER SINUSOIDAL SOLUTION

We will here show more in detail how to find the second order sinusoidal solution of eq. 15.

The second order wave equation is got by neglecting the third and fourth order terms in eq. 8. The z-dependent equation, i.e. the terms that depend on z, will be

$$-\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}e^{R(z-\eta)}-\frac{\partial\eta}{\partial x}R\,e^{R(z-\eta)}$$
$$+\left[6\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}+\eta\frac{\partial^3\eta}{\partial x^3}\right]e^{R(z-\eta)}+\eta\frac{\partial\eta}{\partial x}R^2e^{R(z-\eta)}$$
$$-\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^3\eta}{\partial x^3}\right]\frac{1}{2}e^{2R(z-\eta)}=0 \qquad (90)$$

which reduces to

$$-\frac{\partial^3\eta}{\partial x^3}\frac{1}{R}-\frac{\partial\eta}{\partial x}R+6\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}+\eta\frac{\partial^3\eta}{\partial x^3}+\eta\frac{\partial\eta}{\partial x}R^2$$
$$-\left[\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^3\eta}{\partial x^3}\right]\frac{1}{2}e^{R(z-\eta)}=0 \qquad (91)$$

The last two terms could cause problems, but this will not happen, neither in higher order theories. The equation could again be divided in a z-independent equation and a z-dependent equation and we must demand

$$\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^3\eta}{\partial x^3}=0 \qquad (92)$$

In eq. 12 we found the first order solution

$$\eta_1=\frac{H}{2}\cos k(x-ct) \qquad (93)$$

which can be used for H/L very small. For H/L not so small $\eta$ will deviate slightly from $\eta_1$ so that we may write

$$\eta=\eta_1+\Delta\eta \qquad (94)$$

where $\Delta\eta$ is of second order magnitude, $o(H/L)^2$, while $\eta$ and $\eta_1$ both are of first order magnitude $o(H/L)$.

Substituting eq. 94 with eq. 93 into eq. 92 we see that only third and fourth order terms remain, and they can all be neglected in a second order theory. So eq. 92 did not give any information about $\Delta\eta$ in eq. 94. Eq. 92 will be fulfilled for any $\Delta\eta$.

Substituting eq. 94 into eq. 91 we get, when neglecting higher order terms,

$$-\frac{\partial^3(\eta_1+\Delta\eta)}{\partial x^3}\frac{1}{R}-\frac{\partial(\eta_1+\Delta\eta)}{\partial x}R+6\frac{\partial\eta_1}{\partial x}\frac{\partial^2\eta_1}{\partial x^2}$$
$$+\eta_1\frac{\partial^3\eta_1}{\partial x^3}+\eta_1\frac{\partial\eta_1}{\partial x}R^2=0 \qquad (95)$$

$\eta_1$, the first order solution, is given in eq. 93. For the first order part of $\eta$ we know from eq. 12 that $R = k = 2\pi/L$. Then eq. 95 will be, with eq. 93

$$-\frac{\partial^3\Delta\eta}{\partial x^3}\frac{1}{R}-\frac{\partial\Delta\eta}{\partial x}R+6\left(\frac{H}{2}\right)^2k^3\sin k(x-ct)\cos k(x-ct) \qquad (96)$$

or

$$\frac{\partial^3\Delta\eta}{\partial x^3}\frac{1}{R}+\frac{\partial\Delta\eta}{\partial x}R=3\left(\frac{H}{2}\right)^2k^3\sin 2k(x-ct) \qquad (97)$$

which has the solution

$$\Delta\eta=\left(\frac{H}{2}\right)^2\frac{k}{2}\cos 2k(x-ct) \qquad (98)$$

with $R = k$. We then got $\Delta\eta = \eta_{2a}$ from eq. 15.

Instead of considering the z-independent equation we can consider eq. 8 for the surface $z = \eta$, which gives the celerity for a second order wave, eq. 14.

The solutions eqs. 16 and 17, with the second order correctoin in the argument of the cosine function can be determined directly from eq. 91. They can also be shown to agree with eq. 15 within second order approximation. Eqs. 17 and 18 give

$$\eta=\frac{H}{2}\cos\left[k(x-ct)+\frac{H}{2}k\sin\theta\right]+\frac{k}{2}\left(\frac{H}{2}\right)^2$$

$$=\frac{H}{2}\cos k(x-ct)\cos\left(\frac{H}{2}k\sin\theta\right)$$

$$-\frac{H}{2}\sin k(x-ct)\sin\left(\frac{H}{2}k\sin\theta\right)+\frac{k}{2}\left(\frac{H}{2}\right)^2 \qquad (99)$$

Using the Maclaurin series for cos and sin we get

$$\eta = \frac{H}{2}\cos k(x-ct)\left[1-\frac{1}{2}\left(\frac{H}{2}k\sin\theta\right)^2+\cdots\right]$$
$$-\frac{H}{2}\sin k(x-ct)\left[\frac{H}{2}k\sin\theta+\cdots\right]+\frac{k}{2}\left(\frac{H}{2}\right)^2 \qquad (100)$$

With $\theta = k\ (x - ct)$ and neglecting terms of third and higher orders this will be

$$\eta = \frac{H}{2}\cos k(x-ct) + \left(\frac{H}{2}\right)^2\frac{k}{2}\cos 2k(x-ct) \qquad (101)$$

which is the same as eq. 15.

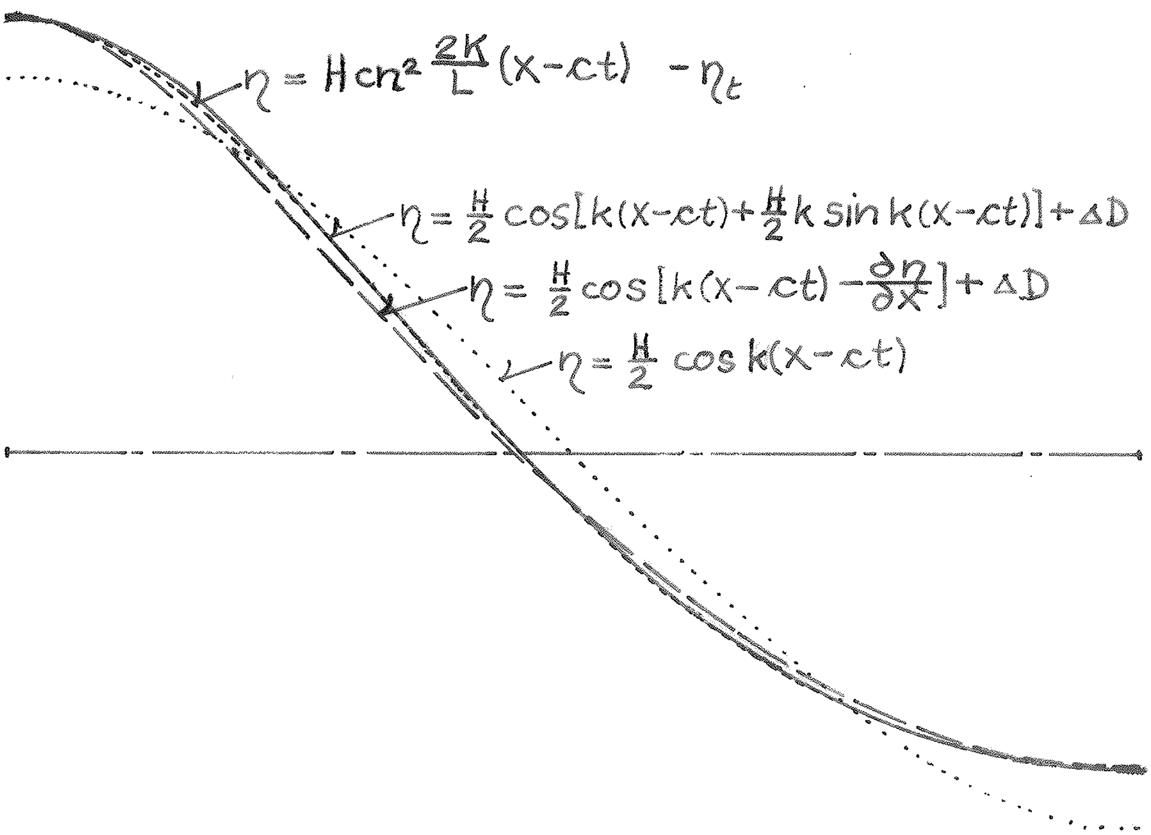



Fig. 13. Comparison of different second order waves for the deep water wave with the steepness H/L = 10%. For infinite deep water the difference is not big. There is though a small difference between the two sinusoidal waves, and this is due to the influence of the third and higher order terms not considered in a second order theory. The second order wave above that is closest to the cnoidal wave is (incidentally) also a third order wave.

APPENDIX VIII

STOKES' THEORY

We will here show that the second order sinusoidal theory for irrotational motion and the Stokes' second order wave are identical within second order approximations.

For the Stokes' wave we have

$$\eta = \frac{H}{2}\cos k(x-ct) + \frac{1}{8}kH^2\cos 2k(x-ct) \tag{102}$$

$$u = \frac{\pi H}{L} c\, e^{kz}\cos k(x-ct) \tag{103}$$

$$w = \frac{\pi H}{L} c\, e^{kz}\sin k(x-ct) \tag{104}$$

$$\frac{p^+}{\gamma} = \frac{H}{2} e^{kz}\cos k(x-ct) - \frac{1}{8}kH^2 e^{2kz} \tag{105}$$

Eq. 102 is seen to be exactly the same as $\eta + \eta_{2a}$ in eq. 15.

$u$ is given in eq. 4, but for a correct comparison with the Stokes' expression we have to use the expression for an irrotational wave, so that eq. 29 demands $\delta = 1/2$ in eq. 28.

$$u_{rot=0} = c\eta k e^{k(z-\eta)} + \frac{1}{2}c\left(\frac{H}{2}\right)^2 k^2 e^{k(z-\eta)} \tag{106}$$

$e^{-k\eta}$ is expanded in the Maclaurin series

$$e^{-k\eta} = 1 - k\eta + \frac{1}{2}k^2\eta^2 + \cdots \tag{107}$$

Then eq. 106 with eq. 15 will be

$$u_{rot=0} = c(\eta_1 + \eta_{2a})k e^{kz} e^{-k\eta} + \frac{1}{2}c\left(\frac{H}{2}\right)^2 k^2 e^{kz} e^{-k\eta} \tag{108}$$

Neglecting third and higher order terms this will be

$$u_{rot=0} = c\eta_1 k e^{kz}(1-k\eta_1) + c\eta_{2a} k e^{kz} + \frac{1}{2}c\left(\frac{H}{2}\right)^2 k^2 e^{kz}$$

$$= c\frac{\pi H}{L} e^{kz}\cos k(x-ct) - c\left(\frac{H}{2}\right)^2 k^2 e^{kz}\cos^2 k(x-ct)$$

$$+ c\frac{1}{2}\left(\frac{H}{2}\right)^2 k^2 e^{kz}\left[\cos 2k(x-ct) + 1\right] \tag{109}$$

which reduces to eq. 103.

But by numerical examples there are some difference between eq. 1o3 and 1o6. This shows the importance of the neglected third order terms even for a deep water wave. ( For H/L = o.1o we find the following velocities at the surface of the crest : $u_{s,c}/c$ = o.45 after Stokes, and $u_{s,c}/c$ = o.41 after eq. 1o6 ).

The vertical velocity is given by eq. 5, which can be written

$$w = -c\frac{\partial\eta}{\partial x}e^{kz}e^{-k\eta} + c\eta\frac{\partial\eta}{\partial x}ke^{kz}e^{-k\eta} \tag{110}$$

With eq. 1o7 this gives neglecting third and higher order terms

$$\begin{aligned} w &= -c\frac{\partial\eta}{\partial x}e^{kz} + c\frac{\partial\eta}{\partial x}k\eta e^{kz} + c\eta\frac{\partial\eta}{\partial x}ke^{kz} \\ &= -c\frac{\partial\eta_1}{\partial x}e^{kz} - c\frac{\partial\eta_{2a}}{\partial x}e^{kz} + 2c\frac{\partial\eta_1}{\partial x}k\eta_1 e^{kz} \end{aligned} \tag{111}$$

which by eq. 15 is seen to be the same as eq. 1o4.

For the pressure we use eq. 73

$$\frac{p^+}{\gamma} = \frac{p}{\gamma} + z = \eta e^{k(z-\eta)} + \frac{\pi}{4}H\frac{H}{L}\left[e^{k(z-\eta)} - e^{2k(z-\eta)}\right] \tag{112}$$

which with eq. 1o7 will be, to the second order

$$\frac{p^+}{\gamma} = \eta_1 e^{kz} + \eta_{2a}e^{kz} - \eta_1 k\eta_1 e^{kz} + \frac{1}{2}k\left(\frac{H}{2}\right)^2\left[e^{kz} - e^{2kz}\right] \tag{113}$$

which by eq. 15 is seen to be the same as eq. 1o5.

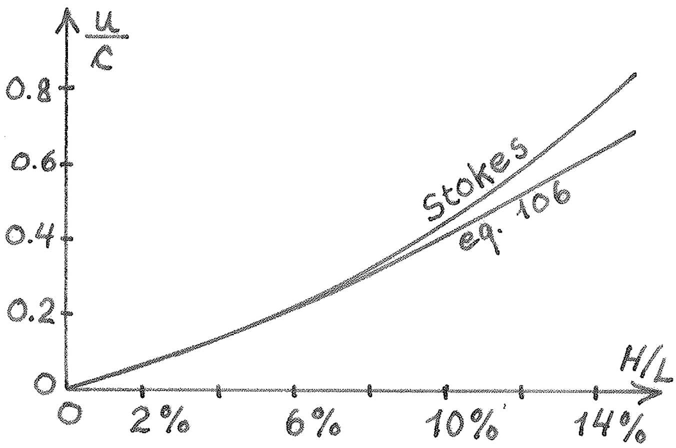



Fig. 14. The horizontal velocity at the surface of the crest for two second order irrotational sinusoidal deep water waves, the Stokes' wave and the second order sinusoidal wave of this chapter.

PRACTICAL APPROACH TO THE CNOIDAL SOLUTION

We will in this appendix discuss how to find a higher order wave solution like the cnoidal wave.

In the first order theory we found a wave equation, eq. 9, which simply led directly to the first order wave solution, eq. 12. By including also the second order terms from eq. 8 it is found natural, with many years of tradition in wave solutions of the perturbation type, to reach the solution of eq. 15. After still some calculations the third order perturbation solution is found in eqs. 45, 46, 47, and 48. Continuing along the same line it is possible after rather much more work to find a fourth order solution or any higher order solution. In this way an apparently very high degree of accuracy can be obtained. But it is known from Stokes' second order wave solution used on more shallow waters that the accuracy does not need to become better. But for deep water this method is applicable.

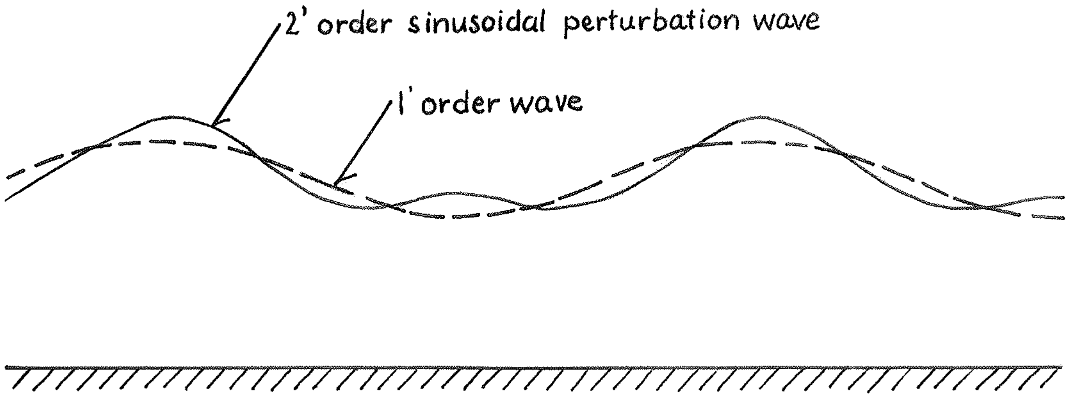



Fig. 15. The first order wave and the second order sinusoidal perturbation wave. (For deep water waves the second order wave will though never get the shown 'crest in the trough'.) The second order wave shows qualitatively how the first order profile should be changed, but a more reasonable solution than the perturbation solution should be tried.

Still it is not completely satisfactory to have a $n^{th}$ order solution that (in the surface profile) 'waves' or 'winds' around the wanted more 'smooth' solution (the real regular wave). It would be nicer to be able to go right from the first order solution to the 'smooth' solution. For this purpose it is obvious to think of the cnoidal solution, which has a long succesful tradition in shallow waters. But such a solution does not seem straight forward.

Traditional cnoidal waves are all over tightly connected with the shallow water criterion. And when considering the Fourier series for the cn or $cn^2$ function it is found not to contain the second harmonic wave, so it may seem very difficult to find a cnoidal wave.

The sinusoidal perturbation solutions are of big value in describing the qualitative changes that should be made on the first order wave. The second order solution of eq. 15 shows that the crest should be higher and narrower and the trough less deep and wider. The third order solution shows that the celerity should be bigger, eq. 52. So wanting to make a better second order solution we need to consider the surface profile.

We would then like to include such changes in our expression for the surface elevation that the first order wave profile changes its shape to give the wanted form of a narrow crest and a wide trough.

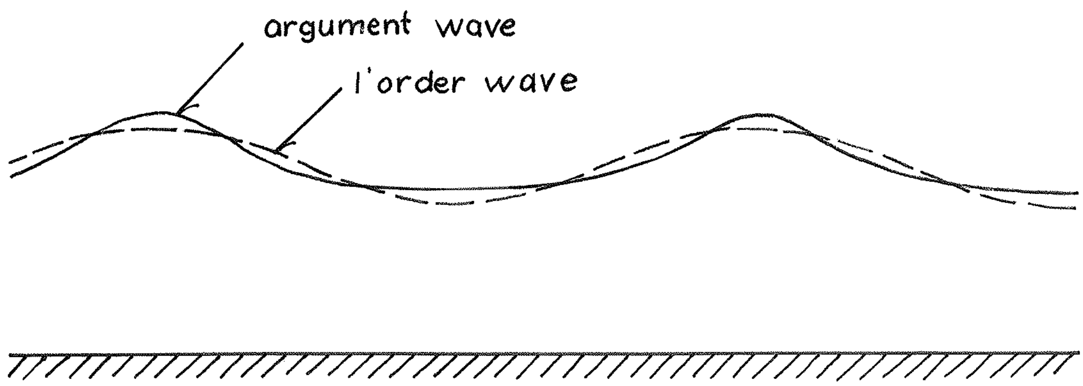
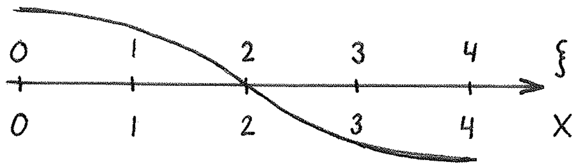



Fig. 16. By including the second order correction in the argument of the cosine function, the wave profile changes as wanted. But big changes cannot be made in this way.

To obtain this, we would like the horizontal x-axis to 'shrink' at the crest and to 'expand' at the trough. An obvious way to get that, is by changing the argument as shown in the solutions eq. 16 or 17. Then the form of the wave will be reasonable if the wave is not too high.

But for high waves on more shallow waters a solution of this type is insufficient. We only need to consider eq. 16 for the trough. A very wide trough will give $\partial\eta/\partial x$ close to o. This means that the argument in eq. 16 is not changed nearly as much as needed to get the wanted very wide trough.

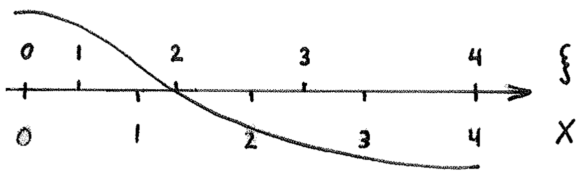
$$\eta = \frac{H}{2} \cos \frac{2\pi}{L} x \quad \text{or} \quad \eta = H \left[ \cos^2 \frac{\pi}{L} x - \frac{1}{2} \right]$$




$$\xi = x$$

Fig. 17 Let us consider a wave with L/2 = 4 and see how we can change its shape. $\eta$ is here a normal cosine function of both x and $\xi$.

$$\eta = \frac{H}{2} \cos \frac{2\pi}{L} \xi$$



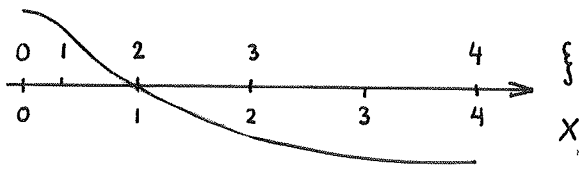

$$\xi = x + \alpha \frac{H}{2} \sin \frac{2\pi}{L} x$$

or

$$\xi = x - \alpha \frac{\partial \eta}{\partial x} \frac{L}{2\pi}$$

Fig. 18. $\eta$ is here a normal cosine function of $\xi$, but $\xi$ is now a function of x, which distorts the surface profile to give the wanted more narrow crest.

$$\eta = H \left[ \cos^2 \frac{\pi}{L} \xi - \frac{1}{2} \right] = H \left[ cn^2 \frac{2K}{L} x - \frac{1}{2} \right]$$




$$\frac{\pi}{L} \xi = am \frac{2K}{L} x$$

$$\frac{2K}{L} x = F \left( \frac{\pi}{L} \xi \right)$$

Fig. 19. $\eta$ is still a cosine function of $\xi$, but $\xi$ is an elliptic function of x, which distorts the surface profile as wanted. The result is the cnoidal wave.

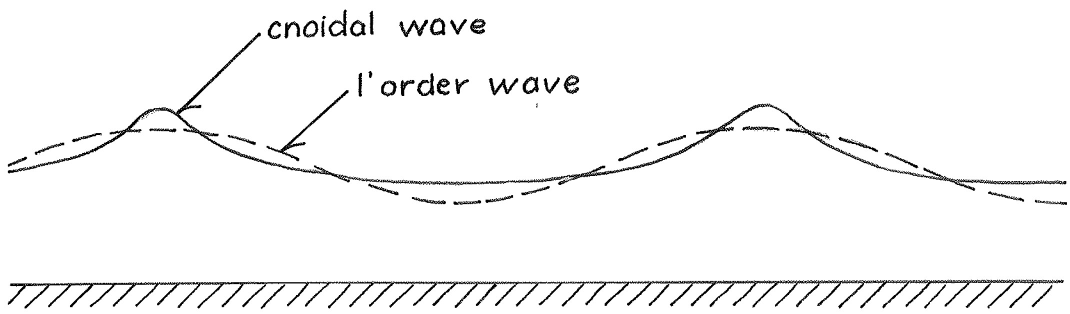



Fig. 20. If the argument of the cosine function is changed by an elliptic function to give the cnoidal wave, the changes of the surface profile can be made as big as wanted, and the wave profile will always have a reasonable form.

A better way to change the argument of the cosine function is to use elliptic functions. The variable $\xi_1$ is a function of $x_1$

$$\xi_1 = \mathrm{am}\ x_1 \tag{114}$$

where am is the Jacobian amplitude. Then we have

$$x_1 = F\ (\xi_1) \tag{115}$$

where F is the incomplete elliptic integral of the first kind. F depends also on the parameter m. F can be found tabulated in many handbooks. We then get the Jacobian elliptic cosine function as

$$\cos\ \xi_1 = \cos\ (\ \mathrm{am}\ x_1\ ) = \mathrm{cn}\ x_1 \tag{116}$$

But cn $x_1$ cannot be used because it makes the x-axis 'shrink' both at the crest and at the trough. This problem is overcome simply by using the square, $\mathrm{cn}^2\ x_1$. We then end up with a solution as eq. 21.

For the cnoidal wave of this thesis it was possible to use the hydrodynamic conditions to make a mathematical theory which would end with a differential equation that had the solution of eq. 21. This is how wave theories usually are made, also the traditional cnoidal theory. But the author finds it more practical for the engineer to use observations from the nature and the laboratory and to use the above considerations to reach the solution directly, and then afterwards find under what conditions it can be used ( eqs. 25 and 26 ).

# APPENDIX X

## FORMULAS FOR THE PROGRESSIVE DEEP WATER WAVE

First order formulas are given in chapter II

## SECOND ORDER SINUSOIDAL WAVE.

$$\eta=\eta_1+\eta_{2a}=\frac{H}{2}\cos k(x-ct)+\left(\frac{H}{2}\right)^2\frac{k}{2}\cos 2k(x-ct) \qquad (117)$$

or

$$\eta=\frac{H}{2}\cos\left[k(x-ct)+\frac{H}{2}k\sin k(x-ct)\right]+\frac{\pi}{4}H\frac{H}{L} \qquad (118)$$

$$k=\frac{2\pi}{L} \qquad (119)$$

$$c=\sqrt{\frac{gL}{2\pi}}=\sqrt{\frac{g}{k}} \qquad (120)$$

$$L=cT \quad \text{so} \quad L=\frac{g}{2\pi}T^2 \qquad (121)$$

$$u=ck\eta e^{k(z-\eta)} \qquad (122)$$

$$w=c\frac{\partial\eta}{\partial x}\left[-1+k\eta\right]e^{k(z-\eta)} \qquad (123)$$

$$\frac{p}{\gamma}+z=\eta e^{k(z-\eta)}+\frac{\pi}{4}H\frac{H}{L}e^{k(z-\eta)}\left[1-e^{k(z-\eta)}\right] \qquad (124)$$

$$\frac{\partial\eta}{\partial x}=-\frac{H}{2}k\sin k(x-ct)-\left(\frac{H}{2}\right)^2k^2\sin 2k(x-ct) \qquad (125)$$

$$\frac{du}{dt}=ckw \qquad (126)$$

## ROTATIONAL SECOND ORDER WAVES

For waves with rotation of only second order magnitude, only u will be changed by rotation

$$u_{rot}=ck\eta e^{k(z-\eta)}+\delta c\left(\frac{H}{2}\right)^2k^2e^{k(z-\eta)}+c\left(\frac{H}{2}\right)^2F(z) \qquad (127)$$

$$\Omega=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\left(\frac{H}{2}\right)^2ck^3e^{k(z-\eta)}\left[\delta-\frac{1}{2}\right]+c\left(\frac{H}{2}\right)^2\frac{\partial F(z)}{\partial z} \qquad (128)$$

Second order waves with rotation of first order magnitude.

$$\eta = \eta_1 + \eta_{2a} + \sum(\eta_{2ci} + \eta_{2di}) \tag{129}$$

$$\eta_{2ci} = \left(\frac{H}{2}\right)^2 k\,\delta_i n_i \cos k(x-ct) \tag{130}$$

$$\eta_{2di} = -\left(\frac{H}{2}\right)^2 k\,\delta_i \frac{n_i^2}{n_i+2} \cos k(x-ct) \tag{131}$$

$$c = \sqrt{\frac{g}{k}}\left[1 + Hk\sum\frac{\delta_i n_i}{n_i+2}\right] \tag{132}$$

$$\begin{aligned} u = {} & c(\eta_1+\eta_{2a})\,k\,e^{k(z-\eta)} \\ & + c\sum \eta_{2ci}\, n_i k\, e^{n_i k(z-\eta)} \\ & + c\sum \eta_{2di}\,(n_i+1)k\, e^{(n_i+1)k(z-\eta)} \\ & + c\frac{H}{2}k\sum \delta_i n_i\, e^{n_i k(z-\eta)} \end{aligned} \tag{133}$$

$$\begin{aligned} w = {} & -c\left[\frac{\partial\eta_1}{\partial x} + \frac{\partial\eta_{2a}}{\partial x}\right] e^{k(z-\eta)} \\ & - c\sum\frac{\partial\eta_{2ci}}{\partial x}\, e^{n_i k(z-\eta)} \\ & - c\sum\frac{\partial\eta_{2di}}{\partial x}\, e^{(n_i+1)k(z-\eta)} \\ & + c\frac{\partial\eta}{\partial x}\eta k\, e^{k(z-\eta)} \\ & + c\frac{H}{2}\frac{\partial\eta}{\partial x}k\sum\delta_i n_i\, e^{n_i k(z-\eta)} \end{aligned} \tag{134}$$

$$\begin{aligned} \frac{p}{\gamma} + z = {} & (\eta_1+\eta_{2a})\,e^{k(z-\eta)} \\ & + \sum\eta_{2ci}\, e^{n_i k(z-\eta)} \\ & + \sum\eta_{2di}\, e^{(n_i+1)k(z-\eta)} \\ & + \frac{\pi}{4}H\frac{H}{L}\, e^{k(z-\eta)}\left[1 - e^{k(z-\eta)}\right] \end{aligned} \tag{135}$$

Third order sinusoidal wave is given in appendix II.

CNOIDAL DEEP WATER WAVE.

$$\eta = H\,cn^2 \frac{2K}{L}(x - ct) + \eta_t = H\,cn^2\theta + \eta_t \tag{136}$$

$$\eta_t = \frac{H}{m}\left(1 - m - \frac{E}{K}\right) \tag{137}$$

$$mK^2 = \pi^3 \frac{H}{L} \tag{138}$$

$$c = \frac{L}{T} = \sqrt{\frac{gL}{2\pi}} = \sqrt{\frac{g}{k}} \tag{139}$$

$$k = \frac{2\pi}{L} \tag{140}$$

$$L = cT \quad \text{so} \quad L = \frac{g}{2\pi} T^2 \tag{141}$$

$$u = c\eta k e^{k(z-\eta)} \tag{142}$$

$$w = -c\frac{\partial\eta}{\partial x}\left[-1 + \eta k\right] e^{k(z-\eta)} \tag{143}$$

$$\frac{p}{\gamma} + z = \eta e^{k(z-\eta)} + \frac{\pi}{4} H \frac{H}{L} e^{k(z-\eta)}\left[1 - e^{k(z-\eta)}\right] \tag{144}$$

$$\frac{\partial\eta}{\partial t} = -c\frac{\partial\eta}{\partial x} = c\,\frac{4KH}{L}\sqrt{cn^2\theta(1 - cn^2\theta)(1 - m + m\,cn^2\theta)} \tag{145}$$

$$\frac{du}{dt} = ckw \tag{146}$$

| H/L | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 | 0.12 | 0.14 |
|---|---|---|---|---|---|---|---|
| $-\eta_t/H$ | 0.48 | 0.47 | 0.45 | 0.44 | 0.43 | 0.41 | 0.40 |
| K | 1.67 | 1.77 | 1.87 | 1.98 | 2.07 | 2.16 | 2.25 |
| m | 0.22 | 0.39 | 0.52 | 0.62 | 0.70 | 0.75 | 0.80 |

Table for the progressive cnoidal wave.

If rotation wants to be considered for this cnoidal wave, arbitrary rotation of second order magnitude can be included, so that only u will be influenced

$$u_{rot} = c\eta k e^{k(z-\eta)} + \delta c\left(\frac{H}{2}\right)^2 k^2 e^{k(z-\eta)} + c\left(\frac{H}{2}\right)^2 F(z) \tag{147}$$

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \left(\frac{H}{2}\right)^2 c k^3 e^{k(z-\eta)}\left[\delta - \frac{1}{2}\right] + c\left(\frac{H}{2}\right)^2 \frac{\partial F(z)}{\partial z} \tag{148}$$

To get an irrotational wave it is necessarry to use eq. 147 with $\delta = 1/2$. The arbitrary function F(z) and $\delta$ must be chosen so that the magnitude of the terms in eq. 148 is of second order.

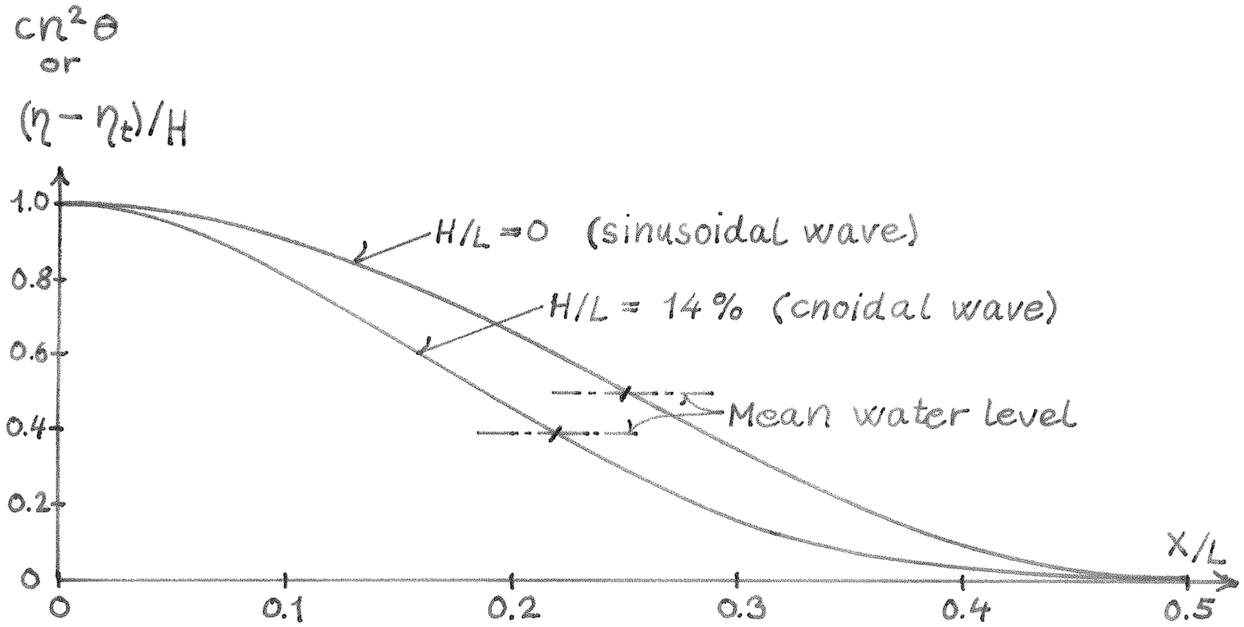



Fig. 21. The wave profile for the progressive cnoidal deep water wave of maximum practical steepness, i.e. H/L = 14%, and the profile of the sinusoidal wave. For all other steepnesses the wave profile will be in between.

APPENDIX XI

NUMERICAL EXAMPLE FOR THE CNOIDAL THEORY

The practical use of the final formulas given in Appendix X will be illustrated by a numerical example, and the results will be compared to those found by the Stokes' theory.

If the wave period and the wave height are

$$T = 10 \text{ sec} \quad ; \quad H = 15.6 \text{ m} \tag{149}$$

the wave length, the steepness, and the celerity will be, using eqs. 141 and 139

$$L = \frac{g}{2\pi} T = 156 \text{ m}; \ \frac{H}{L} = \frac{15.6}{156} = 0.10$$

$$c = \frac{L}{T} = 15.6 \text{ m/sec} \tag{150}$$

The table after eq. 146 gives the crest height and trough depth

$$\eta_c = (1 - 0.43) H = 0.57 \text{ H} = 8.9 \text{ m} \tag{151}$$

$$-\eta_t = 0.43 \text{ H} = 6.7 \text{ m} \tag{152}$$

The horizontal velocity below the crest at the mean water level z = 0 is given by eqs. 142 and 140

$$u = c\eta_c k\, e^{-k\eta_c} = c \times 0.36 \times 0.70 = 0.25c = 3.9 \text{ m/sec} \tag{153}$$

Using eq. 147 with $\delta = 1/2$ and $F(z) = 0$ we get for an irrotational wave

$$u_{rot=0} = 3.9 + 0.5 = 4.4 \text{ m/sec} \tag{154}$$

so that to get $u_{rot} = 4.9$ m/sec it is necessary that $\delta = 1.0$, i.e. a reasonable wind created rotation.

The pressure below the crest at the mean water level z = 0 is given by fig. 9. or eq. 144

$$\frac{p}{\gamma} = \eta_c\, e^{-k\eta_c} + \frac{\pi}{4} H \frac{H}{L} [e^{-k\eta_c} - e^{-2k\eta_c}]$$

$$= 0.40 \text{ H} + 0.02 \text{ H} = 0.42 \text{ H} = 6.5 \text{ m} \tag{155}$$

The vertical velocity at the surface, where $\eta = 0$, i.e. for $z = 0$, may be wanted. Eq. 136 gives

$$cn^2\theta = \frac{\eta}{H} - \frac{\eta_t}{H} = 0.43 \qquad (156)$$

The table gives

$$K = 2.07 \; ; \quad m = 0.70 \qquad (157)$$

Then eqs. 143 and 145 give

$$w = \frac{\partial\eta}{\partial t} = c \times 4 \times 2.07 \times 0.10 \times (0.43(1-0.43)(1-0.70+0.70 \times 0.43)^{\frac{1}{2}}$$

$$= 0.32\, c = 5.0 \text{ m/sec} \qquad (158)$$

This calculation of w also indicates how to calculate du/dt (when needed in the calculation of e.g. wave forces on piles).

The Stokes' theory was considered in appendix VIII. We will here use eqs. 1o2, 1o3, 1o4, and 1o5. Then we get

$$\eta_c = 0.58\, H = 9.0 \text{ m} \qquad (159)$$

u at $z = 0$ below the crest

$$u = 0.31c = 4.9 \text{ m/sec} \qquad (16o)$$

w at $z = 0$ and $\eta = 0$

$$w = 0.31c = 4.9 \text{ m/sec} \qquad (161)$$

p below the crest at $z = 0$, where $p = p^+$

$$\frac{p}{\gamma} = 0.42\, H = 6.5 \text{ m} \qquad (162)$$

There is found a difference in u. But in most cases the difference between the new cnoidal formulas and the Stokes' formulas is very small. This is only good, because for deep water waves the Stokes' theory is found very reasonable.

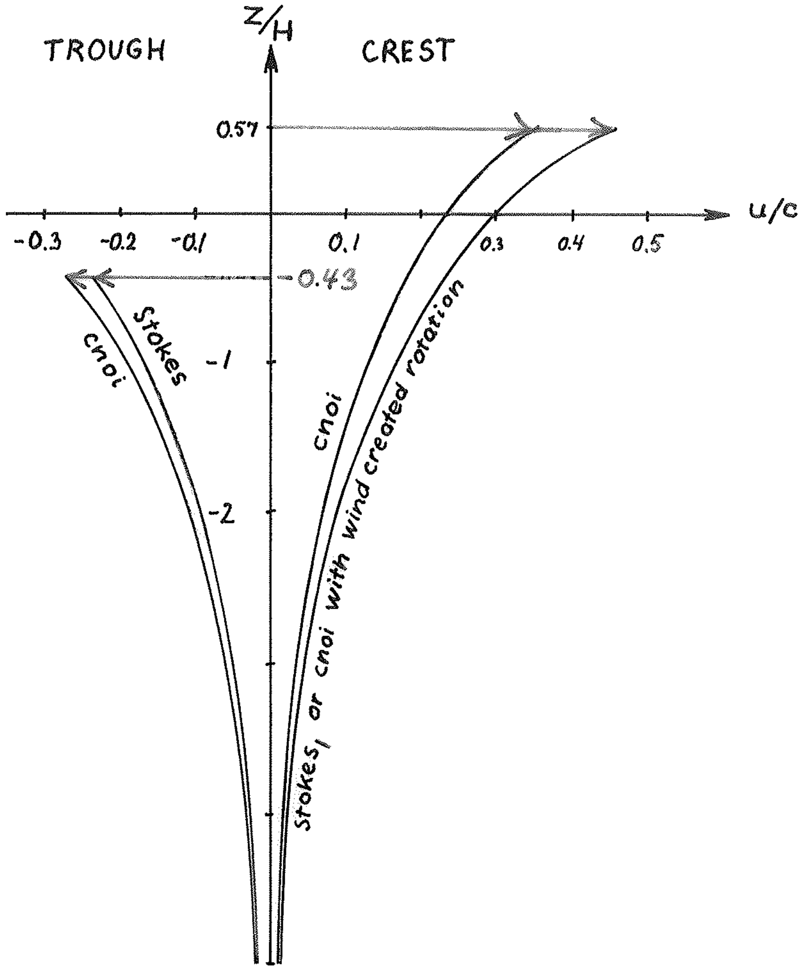



Fig. 22. The horizontal velocity below the crest and the trough in a progressive deep water wave with the steepness H/L = 10%. Comparison between the cnoidal theory and the Stokes' theory. The shown cnoi profiles are from eq. 142. This means that the motion is with a second order 'backward' rotation. The irrotational cnoi profiles will for the crest be halfway between the shown cnoi profile and the Stokes' profile. For the trough the irrotational cnoi profile will be 10% smaller than the Stokes' profile. The Stokes' profile for the crest coincide with the cnoi profile with a 'forward' rotation of a reasonable value ($\delta$ = 1 in eq. 147).

In most cases the difference between the Stokes' wave and the cnoidal wave is small for deep water. But in the next chapters the wave will propagate to waters with arbitrary and even shallow depth. Then it will be found important to use the cnoidal description.